

BPS States, Refined Indices, and Quiver Invariants

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Abstract

For $D = 4$ BPS state construction, counting, and wall-crossing thereof, quiver quantum mechanics offers two alternative approaches, the Coulomb phase and the Higgs phase, which sometimes produce inequivalent counting. The authors have proposed, in arXiv:1205.6511, two conjectures on the precise relationship between the two, with some supporting evidences. Higgs phase ground states are naturally divided into the Intrinsic Higgs sector, which is insensitive to wall-crossings and thus an invariant of quiver, plus a pulled-back ambient cohomology, conjectured to be an one-to-one image of Coulomb phase ground states. In this note, we show that these conjectures hold for all cyclic quivers with Abelian nodes, and further explore angular momentum and R -charge content of individual states. Along the way, we clarify how the protected spin character of BPS states should be computed in the Higgs phase, and further determine the entire Hodge structure of the Higgs phase cohomology. This shows that, while the Coulomb phase states are classified by angular momentum, the Intrinsic Higgs states are classified by R -symmetry.

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1 Introduction

In the study of four-dimensional $N = 2$ supersymmetric theories, one is naturally led to supersymmetric quiver quantum mechanics [1], normalizable ground states of which are images of quantum BPS particle states [2] of the underlying theory in four dimensions. In the language of quiver quantum mechanics, wall-crossings [3, 4] occur when Fayet-Iliopoulos (FI) constants of the quiver change signs, upon which the number of normalizable ground states changes discontinuously. The Coulomb phase realizes a more physical and intuitive picture of this process via multi-centered nature of typical BPS states; some BPS states disappear simply because they are bound states of more than one charge centers and these states become non-normalizable at such crossings [5, 6, 7, 8, 9, 10].

Such discontinuities are also mirrored by the Higgs phase, but in a less intuitive manner. Depending on signs of FI constants, one ends up in different branches, labeled by k in this note, with different vacuum manifolds, say M_k . Each of these branches has its own supersymmetric ground states represented by the cohomology ring, $H(M_k)$ [1]. The relevant index in its simplest form is

$$(-1)^{-d_k} \chi(M_k) = \sum_l (-1)^{l-d_k} \dim H^l(M_k) , \quad (1.1)$$

with $d_k = \dim_{\mathbb{C}} M_k$, whose refined form as well as the Coulomb phase counterpart will be introduced in section 2 and explored thereafter. Although such indices are supposed to be invariant under small continuous deformations, these actually jump discontinuously across marginal stability walls, defined by vanishing of certain FI constants, simply because the respective vacuum manifolds M_k are topologically distinct in different branches.

Since quiver theories in question are quantum mechanics, what we call phases here do not have the same meaning as in field theories. In the latter, with more than two spacetime dimensions, phases imply super-selection sectors, as small fluctuations cannot change vacuum expectation values (vev). In quantum mechanics, there is no such notion as vev, so Higgs and Coulomb phases are just different manifestations or approximations of the same ground state sector physics. Thus, in the absence of some further subtlety, it is natural to expect that Coulomb phase states are mirrored by Higgs phase states and, in one large class of examples, this expectation was confirmed by extensive comparisons [1] and more recently by a general and analytical proof [11].

However, a curious fact is that, when the quiver involves a closed loop, $\chi(M_k)$ sometimes predicts far more numerous Higgs phase ground states than those found in the corresponding Coulomb phase [12]. Upon closer inspections of the quiver dynamics [13, 14], this disparity between the two phases is not entirely surprising, as

the Coulomb phase has some subtleties that could in principle compromise the usual approach; see sections 2 and 7 for more detail. Since (dis-)appearance of BPS states across the walls is clearly due to the multi-centered nature, which is characteristic of the Coulomb phase, these additional states in the Higgs phase are likely unaffected by wall-crossings. On this expectation, it has been suggested that such extra Higgs phase states must have something to do with single-centered black hole solution [15]. Whatever the right spacetime interpretation might be, a more immediate question is how should Higgs phase ground states for general quivers be classified in relation to their Coulomb phase cousins. That is, how to identify the Higgs phase counterpart of the Coulomb phase states, and what are distinguishing geometric and quantum mechanical properties of those extra Higgs phase states that have no Coulomb phase counterpart.

In Ref. [16], the authors proposed two conjectures, starting with the observation that Higgs phase vacuum manifolds, M , are always of the form

$$M = X \Big|_{\partial W=0} \hookrightarrow X, \quad (1.2)$$

where X is a D-term-induced Kähler manifold. W is the superpotential, from which F-term constraints $\partial W = 0$ are generated. Geometrically, this means that M is a complete intersection in the ambient projective variety X . The cohomology of M is naturally split as

$$H(M) = i_M^*(H(X)) \oplus [H(M)/i_M^*(H(X))], \quad (1.3)$$

where $i_M : M \hookrightarrow X$ is the embedding map. Labeling branches by the subscript k as before, the conjectures can be stated as

- The states in the pull-back $i_{M_k}^*(H(X_k))$ of the ambient cohomology are in one-to-one correspondence with the Coulomb phase states. In terms of the equivariant index $\Omega_{\text{Coulomb}}^{(k)}(y)$, we assert that

$$\Omega_{\text{Coulomb}}^{(k)}(y) = (-y)^{-d_k} D_k(-y), \quad (1.4)$$

where, on the right hand side coming from Higgs phase, d_k is the complex dimension of M_k and $D_k(x)$ is the reduced Poincaré polynomial defined as

$$D_k(x) \equiv \sum_l x^l \dim [i_{M_k}^*(H^l(X_k))] . \quad (1.5)$$

- The Intrinsic Higgs states in $H(M_k)/i_{M_k}^*(H(X_k))$ are inherent to the quiver quantum mechanics and insensitive to wall-crossing. In particular, their numerical index, given by

$$(-1)^{-d_k} \chi(M_k) - (-1)^{-d_k} D_k(-1), \quad (1.6)$$

is independent of k , unchanged by wall-crossings.^{#1} More generally, this extends to a refined index of the form

$$(-y)^{-d_k} \chi_{\xi=-y^2}(M_k) - (-y)^{-d_k} D_k(-y) , \quad (1.7)$$

which defines an invariant of the quiver. Here, χ_ξ is the refined Euler character, to be introduced in section 6.

In Ref. [16], an analytical proof of (1.4) was given for cyclic Abelian quivers with three nodes,^{#2} while the invariance of (1.6) was shown analytically for three node cases and at numerical level for four, five, and six node examples. In this note, we wish to work towards complete proofs of the conjectures and explore related issues.

It turns out that, for cyclic Abelian quivers with arbitrary number of nodes, proofs are relatively easy and compact, so those will be the main content of this note. Along the way, we also discuss how the so-called protected spin character [18] of $D = 4$ $N = 2$ supersymmetric theories, carrying both angular momentum and $SU(2)_R$ representations of individual BPS state, should be computed in the Higgs phase. The precise relationship between the protected spin character and the Coulomb phase equivariant index was clarified very recently [13], and here, we extend this to the Higgs phase. Interestingly, we find that the Intrinsic Higgs states, all in singlets of angular momentum $SU(2)_J$, are classified by R-charges.

The organization of this note is as follows. In section 2, we start with a broad overview of refined indices from the underlying four-dimensional theories and discuss how they can be computed in the quiver realization of BPS states. After a brief review of how the protected spin character of the former reduces to the equivariant index of the latter in the Coulomb phase, we propose the Higgs phase version. This in part will justify the refined form of the first conjecture, which is proved later in this note. In section 3, we review basics of the Higgs phase moduli spaces for cyclic Abelian quivers and their cohomologies, in preparation for subsequent discussions. In section 4, we take advantage of the relatively simple form of the reduced Poincaré polynomial $D_k(x)$ and show how it is precisely mapped to the Coulomb phase equivariant index, thereby providing a proof of the first conjecture.

In section 5, we explore a bit more about the pulled-back cohomology $i_{M_k}^*(H(X_k))$ and give a simple geometric realization. Although a fully general proof of the second conjecture at the refined level will be given in section 6, this geometric view allows an intuitive understanding of why the number of states in the Intrinsic Higgs sector is an invariant. As a byproduct, we also find an elementary arithmetic counting formula

^{#1}Since the parity $(-1)^{d_k}$ is independent of the branch choice k , we will sometimes drop this overall sign factor, as was done in Ref. [16]; restoring the sign to the current form is a trivial exercise.

^{#2}For three-node examples, Bena et.al. [17] independently explored the same phenomena.

for the degeneracy, $D_k(-1) = (-1)^{d_k} \Omega_{\text{Coulomb}}^{(k)}(1)$. In section 6, we come back to the refined index of the Higgs phase as proposed in section 2. A general computing algorithm for the refined index is presented and its properties are illustrated with diverse examples. Thanks to special nature of both Higgs and Coulomb phase states, these refined indices allow us to determine the entire Hodge structure of the cohomology, in particular. As expected on general grounds, we find typically very large degeneracy of the Intrinsic Higgs sector. Finally, the conjectured invariance of the Intrinsic Higgs sector at such refined level is established rigorously. In section 7, we close with comments and further questions.

2 Protected Spin Character, Symmetries, and Refined Indices

Before we start discussing the conjectures, their proofs, and consequences thereof, let us briefly recall the definitions of the relevant refined indices. This will also provide a strong motivation for the two conjectures outlined in the introduction and, in particular, show how the refined indices are to be identified across the two phases.

Those states that we wish to count are particle-like BPS states of four-dimensional $N = 2$ theories, be they Seiberg-Witten, supergravity, or compactified superstring theories. Since $N = 2$ supersymmetry comes with $SU(2)_R \times U(1)_R$ R -symmetry group, one might naturally expect $SU(2)_J \times SU(2)_R \times U(1)_R$ to be the relevant global symmetry for a given particle-like state, where $SU(2)_J$ is the spatial rotation. Of these, however, $U(1)_R$ is “spontaneously broken” by the phase of the relevant central charge, so BPS states would be in a representation under $SU(2)_J \times SU(2)_R$ as

$$[1/2 \text{ hyper}] \otimes (J, I) , \quad (2.1)$$

where the first, universal factor consists of a single $SU(2)_J$ doublet and a pair of singlets. J and I in the second factor denote the spin and the iso-spin representations, respectively, under $SU(2)_J$ and $SU(2)_R$.

The simplest index for BPS states in $N = 2$ theories is the second helicity trace

$$\Omega_2 = -\frac{1}{2} \text{Tr} \, (-1)^{2J_3} (2J_3)^2 , \quad (2.2)$$

where the trace is over the one-particle Hilbert space of a given charge. This simplifies to

$$\Omega_2 = \text{tr} \, (-1)^{2J_3} , \quad (2.3)$$

when we factor out the universal “center of mass” degrees of freedom. In practice, this is reflected on the lowercase “tr” which ignores the 1/2 hyper part. As is clear

from this, a half-hyper multiplet would contribute $+1$ and a vector multiplet, -2 , and so on. More generally, this index can be elevated to the protected spin character [18], as

$$\Omega(y) = \text{tr} \left((-1)^{2J_3} y^{2J_3+2I_3} \right), \quad (2.4)$$

which ultimately is the object that we wish to compute and compare in the Higgs and the Coulomb phases.

The story of how this protected spin character reduces to refined indices of quiver quantum mechanics can be surprisingly subtle. For the Coulomb phase, where the bi-fundamental chiral fields are integrated out, the remaining degrees of freedom are the position three-vectors \vec{x}_i of the $n+1$ charge centers and the fermionic partners thereof, four real for each i . In this phase, it has been shown that there is actually an $SO(4)$ global symmetry [14, 13]. The three-vectors and the fermionic partners are in $(\mathbf{3}, \mathbf{1})$ and $(\mathbf{2}, \mathbf{2})$, respectively, under $SO(4) = SU(2)_J \times SU(2)_R$. As such, the supercharges are also in $(\mathbf{2}, \mathbf{2})$ and the protected spin character descends to the $\mathbb{R}^{3(n+1)}$ Coulomb phase quantum mechanics, verbatim,

$$\Omega_{\text{Coulomb}}(y) = \text{tr} \left((-1)^{2J_3} y^{2J_3+2I_3} \right). \quad (2.5)$$

The manifestation of $SU(2)_J \times SU(2)_R$ symmetry in the Coulomb phase is related to the fact that this picture can be directly derived from low energy dynamics of quantum solitons (or black holes) in four-dimensional $N=2$ theories [14].

A more subtle step, which had caused some confusions in literature in the past, occurs when one tries to formulate this index problem in terms of the classical moduli space determined by Denef's formulae,

$$\zeta_i = \sum_{j \neq i} \frac{\langle \gamma_i, \gamma_j \rangle}{|\vec{x}_i - \vec{x}_j|}, \quad (2.6)$$

which carve out a $2n$ -dimensional submanifold in the “relative position space” \mathbb{R}^{3n} spanned by $\vec{x}_i - \vec{x}_j$'s. As will be commented upon later, ζ_i 's are the FI constants of the quiver theory, while the numerators on the right hand side are the linking numbers which count bi-fundamental fields, or equivalently the Schwinger products of the charge pairs. Although it appears natural to reduce dynamics onto such a classical moduli space, no such consistent truncation exists in the quiver theory in general. The main problem is that quantum gaps along classical moduli space are always equal to the gaps along the classically massive directions [13]. This is one main qualitative difference between the Coulomb phase dynamics and the Higgs phase dynamics, as no such subtlety shows in the Higgs phase.

Instead, one may deform the dynamics by hand to make the classical gaps along “radial” directions, $|\vec{x}_i - \vec{x}_j|$, to be much larger than the quantum gaps along the

classically flat “angular” directions. This deformation preserves one out of four supercharges, say \mathcal{Q} , and also preserves the diagonal combination of $SU(2)_J \times SU(2)_R$. Calling the latter $SU(2)_{\mathcal{J}}$ with $\mathcal{J} = J + I$, it has been shown [13] that the protected spin character thus reduces to the refined index of the form^{#3}

$$\Omega_{\text{Coulomb}}(y) = (-1)^{-n + \sum_{i>j} \langle \gamma_i, \gamma_j \rangle} \text{tr} (-1)^F y^{2\mathcal{J}_3} . \quad (2.7)$$

Here, the chirality operator $(-1)^F$, which anticommutes with the surviving supercharge \mathcal{Q} , can be thought of as $(-1)^{2I_3}$; although $SU(2)_R$ is no longer a symmetry, its discrete remnant $(-1)^{2I_3}$ still is. Furthermore, this chirality operator coincides with the canonical choice in mathematics literature, with which standard index theorems are stated. The resulting index is what has been identified as the equivariant index of the Coulomb phase, $\Omega_{\text{Coulomb}}(y)$. See Refs. [7, 19, 1, 12, 20, 21, 14, 15, 13] for physical derivations and computations, in one limit or another.

An important consistency check of the above refined index is that $y^{2\mathcal{J}_3}$ should commute with the surviving supercharge. In other words, there should be at least one supercharge, \mathcal{Q} , such that \mathcal{Q}^2 can act as the Hamiltonian and

$$\{(-1)^F y^{2\mathcal{J}_3}, \mathcal{Q}\} = 0 . \quad (2.8)$$

The latter is possible only if $[\mathcal{J}_3, \mathcal{Q}] = 0$, since y is arbitrary. Otherwise, the above refined index itself makes no sense. In the Coulomb phase dynamics, this comes about because the supercharges in **(2, 2)** under $SU(2)_J \times SU(2)_R$ decompose into **3** + **1** under the diagonal $SU(2)_{\mathcal{J}}$, and because the deformation preserves precisely the singlet supercharge, \mathcal{Q} , from which $[\mathcal{J}_3, \mathcal{Q}] = 0$ follows [13].

Before moving on to the Higgs phase, let us note that all BPS states constructed to date from the multi-center picture are $SU(2)_R$ singlets.^{#4} Although the refined Coulomb phase index is defined with the grading $2\mathcal{J}_3$, this observation leads us to believe that $y^{2\mathcal{J}_3}$ is effectively the same as y^{2J_3} in the supersymmetric ground state sector. It may as well be that $\Omega_{\text{Coulomb}}(y)$ simply keeps track of angular momentum content of Coulomb phase BPS states [18]. Nevertheless, it is important to remember that the refined index should be defined with the $y^{2\mathcal{J}_3}$ insertion; otherwise, the expression would not reduce to the ground state sector and thus will not define an index.

For the Higgs phase expression of the protected spin character, first recall that, there, the dynamics is a supersymmetric nonlinear sigma model onto a complete

^{#3}Strictly speaking, the overall sign in this formula is valid for Abelian quivers only. Its generalization to cases with more than one identical particles is rather involved. We refer readers to [13, 21, 15].

^{#4}For an extensive list of examples from explicit field theory constructions, see Ref. [22].

intersection $M \hookrightarrow X$ inside a projective variety X . In contrast to the Coulomb phase, reduction of dynamics down to the classical moduli space is straightforward as long as FI constants are of large enough absolute values. M inherits the Kählerian properties from the quiver data and is automatically equipped with $SU(2)_{\text{Lefschetz}}$ symmetry acting on wavefunctions. The three generators of $SU(2)_{\text{Lefschetz}}$ can be represented as [23]

$$L_3 = (l - d)/2, \quad L_+ = K \wedge, \quad L_- = K \lrcorner, \quad (2.9)$$

when acting on the cohomology $H(M) = \bigoplus_l H^l(M)$. Here, $d = \dim_{\mathbb{C}} M$ is the complex dimension of the vacuum manifold, as before, and K is the Kähler two-form.

In the past, the grading by $2L_3$ in the Higgs phase index, manifest in the Poincaré polynomial of M for example, has been successfully matched with the grading by $2\mathcal{J}_3$ in $\Omega_{\text{Coulomb}}(y)$ [1, 21, 11]. On the other hand, since Ω_{Coulomb} 's that have been so far used in such comparisons contain only $SU(2)_R$ singlet states, this leaves an ambiguity of whether $L = \mathcal{J}$ or $L = J$. What fixes the matter once and for all is how supercharges transform under $SU(2)_{\text{Lefschetz}}$; as we will see below, from Eq. (2.15), it is clear that y^{2L_3} cannot commute with any of the four supercharges. This shows that $SU(2)_{\text{Lefschetz}} = SU(2)_J$ and $L = J$; We must find an analog of \mathcal{J}_3 , or equivalently an analog of I_3 , to compute the protected spin character in the Higgs phase.

Recall that all quiver theories are equipped with a $U(1)'_R$ symmetry, to be distinguished from $U(1)_R$ of the underlying four-dimensional $N = 2$ theories. This symmetry is already evident in and inherited from $D = 4$ version of the quiver theories, so $U(1)'_R$ clearly commutes with the spatial rotation group $SU(2)_J$. Upon the Hodge decomposition of differential forms, and in particular, of the ground state sector,

$$H(M) = \bigoplus_l H^l(M) = \bigoplus_{p,q} H^{p,q}(M), \quad (2.10)$$

the obvious $U(1)'_R$ charge assignment is

$$\mathcal{I} = (p - q)/2, \quad (2.11)$$

on $H^{p,q}$, from which it immediately follows $[\mathcal{I}, L_{1,2,3}] = 0$. With $SU(2)_{\text{Lefschetz}} = SU(2)_J$, $U(1)'_R$ generated by \mathcal{I} must then be a remnant of $SU(2)_R$ of the underlying four-dimensional $N = 2$ supersymmetric theories. This leads us to propose that the protected spin character is computed in the Higgs phase as

$$\begin{aligned} \Omega_{\text{Higgs}}(y) &= \text{tr} (-1)^{2L_3} y^{2L_3 + 2\mathcal{I}} \\ &= \text{tr} (-1)^{l-d} y^{l-d+p-q} \\ &= \text{tr} (-1)^{p+q-d} y^{2p-d}, \end{aligned} \quad (2.12)$$

where for the last equality we used $l = p + q$.

Again, an important consistency check here is the existence of a supercharge, \mathcal{Q} (in the Higgs phase, together with its Hermitian conjugate \mathcal{Q}^\dagger), which commutes with $L_3 + \mathcal{I}$. For this, let us divide the complex fermions into holomorphic ϕ^α , anti-holomorphic $\bar{\chi}^{\bar{\beta}}$, and their conjugates ϕ^\dagger_α and $\bar{\chi}^\dagger_{\bar{\beta}}$, respectively. The canonical supercharge is schematically,

$$\mathcal{Q} \sim \phi^\alpha \partial_\alpha + \bar{\chi}^{\bar{\beta}} \partial_{\bar{\beta}} . \quad (2.13)$$

Treating ϕ and $\bar{\chi}$ as the creation operators among fermions and adopting the usual differential form representation of states, we may identify

$$\mathcal{Q} = \partial + \bar{\partial} , \quad (2.14)$$

where ∂ ($\bar{\partial}$) is the (anti-)holomorphic exterior derivative. In this language, we have

$$L_3 = \frac{1}{2} (\phi\phi^\dagger + \chi\chi^\dagger - d) , \quad \mathcal{I} = \frac{1}{2} (\phi\phi^\dagger - \chi\chi^\dagger) , \quad (2.15)$$

which shows that $L_3 + \mathcal{I} = \phi\phi^\dagger - d/2$ commutes with both $\bar{\partial}$ and its adjoint $\bar{\partial}^\dagger$. Therefore, we have a complex supercharge pair $\mathcal{Q} = \bar{\partial}$ and $\mathcal{Q}^\dagger = \bar{\partial}^\dagger$, with respect to which $\Omega_{\text{Higgs}}(y)$ is a refined index.

Finally, recall how our conjectures naturally split the cohomology of M in the Higgs phase into the Intrinsic Higgs sector and the pull-back $i_M^*(H(X))$ of the ambient cohomology. In all of our examples, the ambient space X has a simple cohomology structure

$$H(X) = \bigoplus_p H^{p,p}(X) , \quad (2.16)$$

corresponding to the statement that $i_M^*(H(X))$ contains only $SU(2)_R$ singlets (or $U(1)'_R$ neutral). Then, the pulled-back part of the Higgs phase index can be expressed as

$$\begin{aligned} \Omega_{\text{Higgs}}(y) \Big|_{i_M^*(H(X))} &= \text{tr}_{i_M^*(H(X))} (-1)^{2L_3} y^{2L_3} \\ &= (-y)^{-d} \text{tr}_{i_M^*(H(X))} (-y)^l \\ &= (-y)^{-d} D(-y) . \end{aligned} \quad (2.17)$$

Here, in the last step, the reduced Poincaré polynomial $D(x)$, defined in Eq. (1.5), appears, motivating the refined version of the first conjecture as stated in Eq. (1.4). The Intrinsic Higgs sector, on the other hand, is constrained to the middle cohomology

by the Lefschetz hyperplane theorem [16, 23] and the refined index restricted there becomes

$$\begin{aligned} \Omega_{\text{Higgs}}(y) \Big|_{\text{Intrinsic}} &= \text{tr}_{H(M)/i_M^*(H(X))} y^{2\mathcal{I}} \\ &= \text{tr}_{H(M)/i_M^*(H(X))} y^{2p-d} . \end{aligned} \quad (2.18)$$

The Intrinsic Higgs sector belongs to the middle cohomology $p + q = d$, i.e., states in this sector are all angular momentum singlets. This restriction to the middle cohomology is valid for irreducible cases where $M \hookrightarrow X$ does not factorize as $(M' \hookrightarrow X') \times Y$. Generalization to reducible cases is straightforward.

3 Higgs Phase Cohomologies: Cyclic $(n + 1)$ -Gon Quivers

Let us start with a cyclic $(n + 1)$ -gon quiver and denote by Z_i the bi-fundamental fields connecting i -th and $(i + 1)$ -th nodes, for $i = 1, \dots, n$. The last, connecting $(n + 1)$ -th node to the first, is called Z_{n+1} . For each i , there are a_i arrows from the i -th node to the $(i + 1)$ -th node, encoding the fact that Z_i is an a_i -dimensional complex vector,

$$Z_i = (Z_i^{(1)}, \dots, Z_i^{(a_i)}) , \quad (3.1)$$

which has charge $(-1, 1)$ with respect to the two $U(1)$'s. See Figure 3.1. Because the quiver has a loop, with linking numbers of the same sign, we should expect a generic superpotential of the type,

$$W(Z_1, Z_2, \dots, Z_{n+1}) = \sum_{\beta_1=1}^{a_1} \cdots \sum_{\beta_{n+1}=1}^{a_{n+1}} c_{\beta_1 \beta_2 \cdots \beta_{n+1}} Z_1^{(\beta_1)} Z_2^{(\beta_2)} \cdots Z_{n+1}^{(\beta_{n+1})} , \quad (3.2)$$

whose F-term vacuum conditions are

$$\partial_{Z_i^{(\beta_i)}} W = 0 \quad (\beta_i = 1, 2, 3, \dots, a_i) , \quad (3.3)$$

for $i = 1, \dots, n + 1$. As argued in Refs. [12, 16], solutions to $\partial W = 0$ split into branches, where one of the complex vectors Z_i 's is identically zero.

The choice of branch is dictated by the D-term constraints, on the other hand. With $U(1)^{n+1}$ gauge groups, of which the overall sum decouples, we have n -independent

D-term conditions

$$\begin{aligned}
|Z_{n+1}|^2 - |Z_1|^2 &= \zeta_1, \\
|Z_1|^2 - |Z_2|^2 &= \zeta_2, \\
|Z_2|^2 - |Z_3|^2 &= \zeta_3, \\
&\vdots \\
|Z_n|^2 - |Z_{n+1}|^2 &= \zeta_{n+1},
\end{aligned} \tag{3.4}$$

with

$$\zeta_1 + \cdots + \zeta_{n+1} = 0. \tag{3.5}$$

The k -th branch is realized when

$$\sum_{i=I}^k \zeta_i > 0, \quad \sum_{i=k+1}^J \zeta_i < 0, \tag{3.6}$$

for consecutive and mutually exclusive sets of I 's and J 's, where the cyclic nature of the indices are understood. In the k -th branch, Z_k is a zero vector and the remaining D-term conditions are solved entirely by

$$X_k = \mathbb{CP}^{a_1-1} \times \cdots \times \mathbb{CP}^{a_{k-1}-1} \times \mathbb{CP}^{a_{k+1}-1} \times \cdots \times \mathbb{CP}^{a_{n+1}-1},$$

where individual sizes of \mathbb{CP}^{a_i-1} do not matter, as long as they are all nonzero. At the boundaries of a branch, one or more of \mathbb{CP} 's get squashed to zero size, and wall-crossing may occur.

Going back to the F-term constraints, we learn that M_k has the form of a complete intersection,

$$M_k = X_k \Big|_{\partial_{Z_k} W=0} \hookrightarrow X_k, \tag{3.7}$$

given by zero-locus of a_k F-terms, $\partial_{Z_k} W = 0$; all other F-term conditions are trivially met by $Z_k = 0$. For M_k in question, we have the complex dimension

$$d_k = \sum_{i \neq k} a_i - a_k - n = \sum_{i=1}^{n+1} a_i - 2a_k - n. \tag{3.8}$$

The Higgs phase ground states are represented by the cohomology $H(M_k)$, and the Poincaré polynomial encodes their counting,

$$P[M_k](x) = \sum_{l=0}^{2d_k} b_l(M_k) x^l \tag{3.9}$$

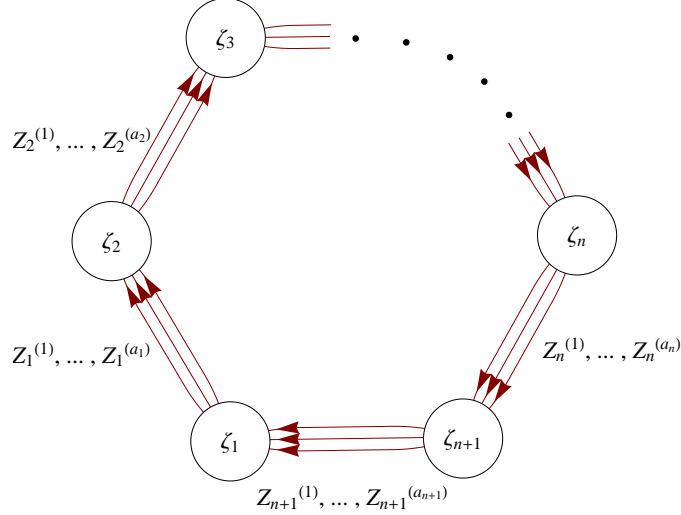


Figure 3.1: The figure, borrowed from Ref. [16], shows a cyclic Abelian quiver with $n + 1$ nodes. Associated with each node is a FI constant ζ_i and a $U(1)$ gauge field. Arrows between i -th and $(i + 1)$ -th nodes represent a_i chiral multiplets of charge $(-1, 1)$, say $Z_i = (Z_i^{(1)}, \dots, Z_i^{(a_i)})$.

with Betti numbers, $b_l(M_k)$. Recalling $SU(2)_{\text{Lefschetz}} = SU(2)_J$, we see that b_l counts the Higgs phase ground states of helicity $(l - d_k)/2$.

Our conjectures split the cohomology $H(M_k)$ into two parts, one of which comes from the pull-back of $H(X_k)$. As used extensively in Ref. [16], the Lefschetz hyperplane theorem implies that the pull-back of $H(X_k)$ is isomorphic to $H(M_k)$ for the lower-half cohomologies of M_k , i.e. up to H^{d_k-1} . Also, the map is injective for H^{d_k} . Combined with the Poincaré duality on M_k , this tells us that

$$H^l(M_k) \simeq H^{2d_k-l}(M_k) \simeq H^l(X_k), \quad l < d_k, \quad (3.10)$$

or equivalently,

$$H(M_k) = i_{M_k}^*(H(X_k)) \oplus [H^{d_k}(M_k)/i_{M_k}^*(H^{d_k}(X_k))]. \quad (3.11)$$

Thus, we learn that the Intrinsic Higgs states all belong to the middle cohomology of M_k .^{#5} Upon the Hodge decomposition, the pull-back occupies the vertical center

^{#5} For more general quivers, we may have reducible cases, where $M = M' \times Y$ and $M' \hookrightarrow X'$ with projective varieties X' and Y . If reducible, we simply have a product structure $H(M) = H(M') \otimes H(Y)$, and much of our discussion from now on should apply to $H(M')$.

line of the Hodge diamond

$$i_{M_k}^*(H(X_k)) \subset \bigoplus_p H^{p,p}(M_k) . \quad (3.12)$$

All other entries, $H^{p,q}(M_k)$ with $p \neq q$ and $p + q \neq d_k$, are null. This generic feature will allow us to determine the individual $H^{p,q}(M_k)$ in section 6.

Full information about $i_{M_k}^*(H(X_k))$ can be found from the well-known Poincaré polynomial of X_k ,

$$P[X_k](x) = \frac{\prod_{i \neq k} (1 - x^{2a_i})}{(1 - x^2)^n} = \sum b_{2l}(X_k) \cdot x^{2l} . \quad (3.13)$$

We truncate this up to the order d_k , and complete it by inverting the lower half to the upper half, which computes the reduced Poincaré polynomial for $M_k \hookrightarrow X_k$,

$$\begin{aligned} D_k(x) &\equiv \sum_l x^l \dim [i_{M_k}^*(H^l(X_k))] \\ &= b_{d_k}(X_k) \cdot x^{d_k} + \sum_{0 \leq 2l < d_k} b_{2l}(X_k) \cdot (x^{2l} + x^{2d_k-2l}) , \end{aligned} \quad (3.14)$$

where $b_{d_k}(X_k)$ is nonvanishing only when d_k is even. Since $i_{M_k}^*(H(X_k))$ belongs to $\bigoplus_p H^{p,p}(M_k)$, this reduced polynomial determines $i_{M_k}^*(H(X_k))$ entirely.

4 Coulomb Inside Higgs: the First Conjecture

In this section, we prove the first conjecture

$$\Omega_{\text{Coulomb}}^{(k)}(y) = (-y)^{-d_k} D_k(-y) , \quad (4.1)$$

for all cyclic Abelian quivers. What facilitates the proof greatly is the observation that only nonpositive-power terms really matter. Recall that the lower half of the reduced Poincaré polynomial $D_k(x)$ is determined entirely by the ambient space X_k cohomology, such that

$$F_k^-(x) \equiv x^{-d_k} D_k(x) \Big|_{\text{nonpositive}} = \left(x^{-d_k} \prod_{i \neq k} \frac{1 - x^{2a_i}}{1 - x^2} \right) \Big|_{\text{nonpositive}} . \quad (4.2)$$

Note that positive-power terms in $x^{-d_k} D_k(x)$ are mirror images of negative-power ones, so $F_k^-(x)$ contains the same information as $D_k(x)$. Similarly $\Omega_{\text{Coulomb}}^{(k)}(y)$ is

symmetric under $y \rightarrow 1/y$, since powers of y are related to helicities, and ultimately to $SU(2)_J$ representations.

The proposed equality, $\Omega_{\text{Coulomb}}^{(k)}(y) = (-y)^{-d_k} D_k(-y)$, is then equivalent to the statement that

$$\Delta_k(y) \equiv \Omega_{\text{Coulomb}}^{(k)}(y) - F_k^-(y) \quad (4.3)$$

contains only positive-power terms of y . In other words, $\Delta_k(0)$ should be well-defined and equal to zero. In turn, the latter is equivalent to the condition that

$$(1 - y^2)^n \cdot \Delta_k(y) = (1 - y^2)^n \cdot \Omega_{\text{Coulomb}}^{(k)}(y) - \left((-y)^{-d_k} \prod_{i \neq k} (1 - y^{2a_i}) \right) \Big|_{\text{nonpositive}} \quad (4.4)$$

vanishes at $y = 0$ and thus contains only positive powers. Thus, with $\Omega_{\text{Coulomb}}^{(k)}$ known, this would serve as a very economical method to prove the first conjecture. Following actual computations of Ω_{Coulomb} , one can see that in general it has the form [15],

$$(-1)^{-n + \sum_{i=1}^{n+1} a_i} \left[\frac{h(y) + (-1)^n h(y^{-1})}{(y - y^{-1})^n} \right], \quad (4.5)$$

where the polynomial h is of definite parity. The condition that Eq. (4.4) contains only positive-power terms demands that non-positive power terms in $y^n h(y^{-1})$ should be in a very particular form determined by the second piece on the right hand side of Eq. (4.4) if the first conjecture is to hold. We will prove this assertion in section 4.2.

Of the polynomial h , only terms with power $\geq n$ enter this comparison, so $\Delta_k(0) = 0$ is clearly a necessary condition to establish the equivalence of the two indices. Not so immediately obvious fact is that this comparison is in fact also sufficient. How can that be? The answer is found in the denominator in Ω_{Coulomb} , which includes a factor $(y - 1)^n$. Unless the numerator also has an n -th order zero at $y = 1$, $\Omega_{\text{Coulomb}}(y)$ would be divergent at $y = 1$, which is physically unacceptable: $\Omega_{\text{Coulomb}}(1)$ is the ground state degeneracy. Splitting the polynomial by the power of y at degree n ,

$$h(y) = h_{\geq}(y) + h_{<}(y), \quad (4.6)$$

the lower part $h_{<}(y)$, a polynomial of degree less than n , is uniquely fixed, once h_{\geq} is given, by the finiteness of $\Omega_{\text{Coulomb}}(1)$. On the other hand, $y^n h_{\geq}(y^{-1})$ is precisely the part to be compared with the non-positive powers of the Higgs phase expression, $(-y)^{-d_k} D_k(-y)$. Since $(-y)^{-d_k} D_k(-y)$ is manifestly regular at $y = 1$ as well, exact matching of $y^n h_{\geq}(y^{-1})$ against its Higgs phase counterpart suffices to establish $\Omega_{\text{Coulomb}}^{(k)}(y) = (-y)^{-d_k} D_k(-y)$.

4.1 Coulomb Phase Equivariant Index

For this, let us first evaluate $\Omega_{\text{Coulomb}}^{(k)}(y)$ directly in the Coulomb phase. Recall that for each branch of the Higgs phase, there is a Coulomb phase counterpart. The moduli space in Coulomb phase is spanned by solutions to multi-particle equilibrium conditions, which, for cyclic $(n+1)$ -gon quivers, are

$$\begin{aligned} \frac{a_{n+1}}{|\vec{x}_{n+1} - \vec{x}_1|} - \frac{a_1}{|\vec{x}_1 - \vec{x}_2|} &= \zeta_1, \\ \frac{a_1}{|\vec{x}_1 - \vec{x}_2|} - \frac{a_2}{|\vec{x}_2 - \vec{x}_3|} &= \zeta_2, \\ &\vdots \\ \frac{a_n}{|\vec{x}_n - \vec{x}_{n+1}|} - \frac{a_{n+1}}{|\vec{x}_{n+1} - \vec{x}_1|} &= \zeta_{n+1}. \end{aligned} \quad (4.7)$$

Three-vectors \vec{x}_i represent position of i -th charge center. This defines a subspace of $\mathbb{R}^{3(n+1)}$. How one obtains the Coulomb phase (equivariant) index from this, for arbitrary number of charge centers, is a long and complicated story [21, 15, 13].

In particular, Refs. [21, 15] developed an extensive technique to compute equivariant indices via a localization method, which is in the end dictated by the fixed point theorem associated with \mathcal{J}_3 ; The fixed points are solutions to (4.7) with all the charge centers located along the z -axis. Taking $z_1 = 0$ to fix the free center of mass position, we find the Coulomb phase equivariant index [21] for our quivers

$$\Omega_{\text{Coulomb}}^{(k)}(y) = \frac{(-1)^{\sum_{i=1}^{n+1} a_i - n}}{(y - y^{-1})^n} \left[\left(\sum_p s(p) y^{\sum_{i=1}^{n+1} a_i \text{sign}[z_i - z_{i+1}]} \right) + H_k(y) + (-1)^n H_k(y^{-1}) \right],$$

$$s(p) = \text{sign}[\det M], \quad (4.8)$$

where the summation in the round brackets is taken over all possible fixed points p and M is an $n \times n$ matrix with the following non-vanishing components:

$$\begin{aligned} M_{i,i} &= a_i \frac{z_i - z_{i+1}}{|z_i - z_{i+1}|^3} + a_{i+1} \frac{z_{i+1} - z_{i+2}}{|z_{i+1} - z_{i+2}|^3}, \\ M_{i,i+1} = M_{i+1,i} &= -a_{i+1} \frac{z_{i+1} - z_{i+2}}{|z_{i+1} - z_{i+2}|^3}. \end{aligned} \quad (4.9)$$

Note that if we have a fixed point with the ordering of charged centers along z -axis as $\{\sigma(1), \sigma(2), \dots, \sigma(n+1)\}$, then correspondingly we will get a fixed point with the reversed ordering of charged centers $\{\sigma(n+1), \dots, \sigma(2), \sigma(1)\}$ by taking all z_i to $-z_i$.

Therefore, we can express the contributions from the fixed points as

$$\sum_p s(p) y^{\sum_{i=1}^{n+1} a_i \text{sign}[z_i - z_{i+1}]} = G_k(y) + (-1)^n G_k(y^{-1}), \quad (4.10)$$

where $G_k(y)$ is a polynomial. Thus, the polynomial h in (4.5) is precisely $h = G_k + H_k$ in branch k .

The H_k terms are added by hand,^{#6} when the quiver admits so-called scaling solutions, when $G(y) + (-1)^n G(y^{-1})$ alone does not have enough zero at $y = 1$ and naively leads to divergent $\Omega_{\text{Coulomb}}(1)$. The prescription of Ref. [15] demands a canceling polynomial of the form

$$H_k(y) = \sum_{\substack{0 \leq l < n \\ l - \sum_{i=1}^{n+1} a_i \in 2\mathbb{Z}}} \lambda_l y^l, \quad (4.11)$$

where the coefficients λ_l are decided uniquely by requiring that $\Omega_{\text{Coulomb}}^{(k)}(y)$ is finite when $y = 1$. Note that the parity of $H_k(y)$ is required to be the same as that of $G_k(y)$; this condition follows from the observation that G_k is of definite parity, for any given quiver, combined with the physical requirement that $\Omega_{\text{Coulomb}}^{(k)}(y)$ itself should have the form $f_k(y) + f_k(y^{-1})$ for some polynomial f_k . These observations suffice to establish the uniqueness of the canceling polynomial H_k in response to the computed G_k .

In order to get an explicit expression of the index for a given quiver, we need first to solve the fixed point equations. For quivers without closed loops, general rules for enumerating fixed points was given in Ref. [11]. However, there is no known systematical method. For more general quivers, let us observe that the index is invariant within each branch, so that we may pick a particularly convenient set of FI constants and simplify the problem. Recalling that the k -th branch is described by

$$\sum_{i=I}^k \zeta_i > 0, \quad \sum_{i=k+1}^J \zeta_i < 0, \quad (4.12)$$

we may compute the index at the following special values of FI constants,

$$\zeta_k = -\zeta_{k+1} > 0, \quad \zeta_i = 0 \quad (i \neq k, k+1), \quad (4.13)$$

where solutions must obey

$$|z_k - z_{k+1}| = \frac{a_k}{\rho}, \quad |z_i - z_{i+1}| = \frac{a_i}{\rho + \zeta_k} \quad (i \neq k),$$

^{#6}Backwardly, our first conjecture which holds with this prescription by Manschot et. al. [15] bolsters the latter as a sensible choice.

$$\sum_{i \neq k} \text{sign}[z_i - z_{i+1}] \frac{a_i}{\rho + \zeta_k} + \text{sign}[z_k - z_{k+1}] \frac{a_k}{\rho} = \sum_i (z_i - z_{i+1}) = 0 . \quad (4.14)$$

For each solution, $\det M$ is

$$\frac{(\rho + \zeta_k)^{2(n-1)}}{\prod_i a_i \text{sign}[z_i - z_{i+1}]} \left(\text{sign}[z_k - z_{k+1}] \cdot a_k (\rho + \zeta_k)^2 + \sum_{i \neq k} \text{sign}[z_i - z_{i+1}] \cdot a_i \rho^2 \right) . \quad (4.15)$$

Since $\rho + \zeta_k > \rho > 0$, if the linking numbers satisfy the inequality $\sum_{i \neq k} a_i t_i < a_k$ with $t_i = \pm 1$, there is no fixed point. On the other hand, if $\sum_{i \neq k} a_i t_i \geq a_k$ holds, one finds a solution for which $s(p)$ is

$$s(p) = \prod_{i \neq k} t_i . \quad (4.16)$$

Therefore, for cyclic $(n+1)$ -gon quivers, we find

$$G_k(y) = \sum_{\{t_{i \neq k} = \pm 1\}} \left[\prod_{i \neq k} t_i \right] \cdot \Theta \left(\sum_{i \neq k} a_i t_i - a_k \right) y^{\sum_{i \neq k} a_i t_i - a_k} , \quad (4.17)$$

where

$$\Theta(x) = \begin{cases} 1 & \text{for } x \geq 0 \\ 0 & \text{for } x < 0 \end{cases} . \quad (4.18)$$

The Coulomb equivariant index for each branch can thus be expressed explicitly, once the subtraction term H_k is (uniquely) determined as aforementioned.

4.2 $\Omega_{\text{Coulomb}}^{(k)}(y) = (-y)^{-d_k} D_k(-y)$

We are now ready to prove the first conjecture for all cyclic Abelian quivers; by comparing the two respective routines (4.8) and (3.14), we will show that $\Omega_{\text{Coulomb}}^{(k)}(y)$ of the Coulomb phase equals $(-y)^{-d_k} D_k(-y)$ of the Higgs phase.

To show this, we will start by rewriting $(-y)^{-d_k} D_k(-y)$ to resemble the Coulomb side. Defining a polynomial \tilde{G}_k as

$$y^n \tilde{G}_k(y^{-1}) \equiv \left(y^{-d_k} \prod_{i \neq k} (1 - y^{2a_i}) \right) \Big|_{\text{nonpositive}} , \quad (4.19)$$

we have,

$$(-y)^{-d_k} D_k(-y)$$

$$= (-1)^{\sum_{i=1}^{n+1} a_i - n} \left[\frac{\tilde{G}_k(y) + (-1)^n \tilde{G}_k(y^{-1}) + \tilde{H}_k(y) + (-1)^n \tilde{H}_k(y^{-1})}{(y - y^{-1})^n} \right], \quad (4.20)$$

where we used $(-1)^{d_k} = (-1)^{\sum_{i=1}^{n+1} a_i - n}$. Unlike the Coulomb phase, the polynomial, \tilde{H}_k , is already fixed by (3.14). Nevertheless, it can be thought of as the Higgs phase analog of H_k , with its highest power less than n , in the sense that, if we drops it, (4.20) would generally become divergent at $y = 1$. Comparing this against the general expression on the Coulomb side,

$$\begin{aligned} & \Omega_{\text{Coulomb}}^{(k)}(y) \\ &= (-1)^{\sum_{i=1}^{n+1} a_i - n} \left[\frac{G_k(y) + (-1)^n G_k(y^{-1}) + H_k(y) + (-1)^n H_k(y^{-1})}{(y - y^{-1})^n} \right], \quad (4.21) \end{aligned}$$

and, remembering earlier discussion at the top of the section, we see that the first conjecture holds if and only if the equality

$$y^n G_k(y^{-1}) \Big|_{\text{nonpositive}} = y^n \tilde{G}_k(y^{-1}) \quad (4.22)$$

holds.

Recall from the previous subsection that uniqueness of H_k , given G_k , was guaranteed by three requirements: regularity of index at $y = 1$, definite parity of G_k , and parity of H_k coinciding with that of G_k . All three requirements apply to \tilde{G}_k, \tilde{H}_k pair, in fact trivially since $D_k(x)$ is always an even polynomial. This means that, with only \tilde{G}_k given as (4.19), one could have recovered \tilde{H}_k indirectly by writing,

$$\tilde{H}_k(y) = \sum_{\substack{0 \leq l < n \\ l - \sum_{i=1}^{n+1} a_i \in 2\mathbb{Z}}} \tilde{\lambda}_l y^l, \quad (4.23)$$

and fixing $\tilde{\lambda}$'s uniquely by demanding the regularity. Thanks to the uniqueness, the resulting polynomial $\tilde{G}_k + \tilde{H}_k$ would be exactly the same as what we finds from the known $D_k(x)$ in (3.14) via (4.20). This implies that (4.22) translates to

$$G_k(y) + H_k(y) = \tilde{G}_k(y) + \tilde{H}_k(y), \quad (4.24)$$

and thus proves $\Omega_{\text{Coulomb}}^{(k)}(y) = (-y)^{-d_k} D_k(-y)$.

It only remains to check Eq. (4.22). Replacing d_k by $\sum_{i \neq k} a_i - a_k - n$ in (4.19) and expanding the product, we easily see that possible exponents are $n + a_k - \sum_{i \neq k} t_i a_i$

with $t_i = \pm 1$; $t_i = -1$ corresponds to a multiplicative factor $-y^{2a_i}$ from the expansion, and this is accompanied by -1 prefactor, represented conveniently by t_i itself. This gives

$$y^n \tilde{G}_k(y^{-1}) = \sum_{\{t_{i \neq k} = \pm 1\}} \left[\prod_{i \neq k} t_i \right] \cdot \Theta \left(\sum_{i \neq k} a_i t_i - a_k - n \right) y^{-\sum_{i \neq k} a_i t_i + a_k + n}, \quad (4.25)$$

whose features are reminiscent of the Coulomb phase expression in Eq. (4.17). Indeed, starting with the latter and flipping $y \rightarrow 1/y$, we find

$$\begin{aligned} y^n G_k(y^{-1}) \Big|_{\text{nonpositive}} &= y^n \sum_{\{t_{i \neq k} = \pm 1\}} \left[\prod_{i \neq k} t_i \right] \cdot \Theta \left(\sum_{i \neq k} a_i t_i - a_k \right) y^{-\sum_{i \neq k} a_i t_i + a_k} \Big|_{\text{nonpositive}} \\ &= \sum_{\{t_{i \neq k} = \pm 1\}} \left[\prod_{i \neq k} t_i \right] \cdot \Theta \left(\sum_{i \neq k} a_i t_i - a_k - n \right) y^{-\sum_{i \neq k} a_i t_i + a_k + n}. \end{aligned} \quad (4.26)$$

This gives $(-y)^{-d_k} D_k(-y) = \Omega_{\text{Coulomb}}^{(k)}(y)$ as promised, and concludes the proof of the first conjecture for all cyclic Abelian quivers.

4.3 Examples

The procedure we adopted to prove the first conjecture for cyclic Abelian quivers offers an interesting and perhaps more economical method to find $\Omega_{\text{Coulomb}}(y)$ without ever going into the Coulomb phase, by computing $\tilde{G}(y) + \tilde{H}(y)$ of the Higgs phase instead of $G(y) + H(y)$ of the Coulomb phase. With $\tilde{G}_k(y)$ given by

$$y^n \tilde{G}_k(y^{-1}) = \left(y^{-d_k} \prod_{i \neq k} (1 - y^{2a_i}) \right) \Big|_{\text{nonpositive}}, \quad (4.27)$$

the simplicity comes from two aspects. First, the right hand side is entirely determined by the ambient manifold X_k ; the complicated F-term conditions enter only via a single integer, d_k . Second, $\prod_{i \neq k} (1 - y^{2a_i}) = (1 - y^2)^n P[X_k](-y)$ contains exactly the same information as $P[X_k](x)$, yet is far less cluttered. Thanks to these features, and the respective uniqueness of H_k and \tilde{H}_k , we can compute the Coulomb phase equivariant index

$$\Omega_{\text{Coulomb}}^{(k)}(y) \leftarrow (-1)^{\sum a_i - n} \times \frac{h_k(y) + (-1)^n h_k(y^{-1})}{(y - y^{-1})^n} \quad (4.28)$$

itself, with $h_k(y) = \tilde{G}_k(y) + \tilde{H}_k(y)$. Below, we exploit this for general 3-gon and 4-gon examples, and display the equivariant indices in their full generality. We suspect that similar simplification will occur for more general class of quivers.

3-Gons

Without loss of generality, let us consider the third branch and compute \tilde{G}_3 using the Higgs phase picture. There are three cases in all.

- Case 1: $a_3 > a_1 + a_2 - 2$

Because the relevant Higgs phase does not exist, we have $\tilde{G}_3(y^{-1}) = 0$, which also means that $\Omega_{\text{Coulomb}}^{(3)}(y) = 0$.

- Case 2: $a_{\sigma(1)} \geq a_{\sigma(2)} + a_3 + 2$ where σ denotes a permutation of $\{1, 2\}$.

In this case, two terms show up in Eq. (4.19) for \tilde{G}_3

$$\tilde{G}_3(y^{-1}) = y^{-a_{\sigma(1)} - a_{\sigma(2)} + a_3} - y^{-a_{\sigma(1)} + a_{\sigma(2)} + a_3}. \quad (4.29)$$

We also notice that no counter terms are needed, $\tilde{H}_3(y) = 0$.

- Case 3: The twisted 3-gon condition is obeyed

$$\begin{aligned} a_1 &< a_2 + a_3 + 2, \\ a_2 &< a_1 + a_3 + 2, \\ a_3 &\leq a_1 + a_2 - 2, \end{aligned} \quad (4.30)$$

whereby only one term exists in \tilde{G}_3 ,

$$\tilde{G}_3(y^{-1}) = y^{-a_1 - a_2 + a_3}, \quad (4.31)$$

and the counter terms are easily determined as

$$\tilde{H}_3(y) = \begin{cases} -y, & \text{if } \sum_i a_i \text{ is odd;} \\ -1, & \text{if } \sum_i a_i \text{ is even.} \end{cases} \quad (4.32)$$

4-Gons

Again without loss of generality, let us consider the fourth branch and determine \tilde{G}_4 . There are six cases in all.

- Case 1: $a_4 > a_1 + a_2 + a_3 - 3$

The corresponding Higgs phase is null, so we find $\tilde{G}_4(y^{-1}) = 0$ and thus $\Omega_{\text{Coulomb}}^{(4)}(y) = 0$.

- Case 2: $a_{\sigma(1)} \geq a_{\sigma(2)} + a_{\sigma(3)} + a_4 + 3$ where σ is a permutation of $\{1, 2, 3\}$.

Following the same procedure, we get from Eq. (4.19)

$$\begin{aligned} \tilde{G}_4(y^{-1}) = & y^{-a_{\sigma(1)}-a_{\sigma(2)}-a_{\sigma(3)}+a_4} - y^{-a_{\sigma(1)}+a_{\sigma(2)}-a_{\sigma(3)}+a_4} \\ & - y^{-a_{\sigma(1)}-a_{\sigma(2)}+a_{\sigma(3)}+a_4} + y^{-a_{\sigma(1)}+a_{\sigma(2)}+a_{\sigma(3)}+a_4}, \end{aligned} \quad (4.33)$$

which requires no counter terms; $\tilde{H}_4(y) = 0$.

- Case 3: The twisted 4-gon condition is obeyed

$$\begin{aligned} a_1 &< a_2 + a_3 + a_4 + 3, \\ a_2 &< a_1 + a_3 + a_4 + 3, \\ a_3 &< a_1 + a_2 + a_4 + 3, \\ a_4 &\leq a_1 + a_2 + a_3 - 3, \end{aligned} \quad (4.34)$$

and in addition

$$\begin{aligned} a_1 + a_2 &\geq a_3 + a_4 + 3, \\ a_1 + a_3 &\geq a_2 + a_4 + 3, \\ a_2 + a_3 &\geq a_1 + a_4 + 3. \end{aligned} \quad (4.35)$$

We find

$$\begin{aligned} \tilde{G}_4(y^{-1}) = & y^{-a_1-a_2-a_3+a_4} - y^{a_1-a_2-a_3+a_4} \\ & - y^{-a_1+a_2-a_3+a_4} - y^{-a_1-a_2+a_3+a_4}, \end{aligned} \quad (4.36)$$

with the counter terms

$$\tilde{H}_4(y) = \begin{cases} -2a_4y, & \text{if } \sum_i a_i \text{ is odd;} \\ -a_4y^2, & \text{if } \sum_i a_i \text{ is even.} \end{cases} \quad (4.37)$$

- Case 4: The twisted 4-gon condition is obeyed and in addition

$$\begin{aligned} a_{\sigma(1)} + a_{\sigma(2)} &\geq a_{\sigma(3)} + a_4 + 3, \\ a_{\sigma(1)} + a_{\sigma(3)} &\geq a_{\sigma(2)} + a_4 + 3, \\ a_{\sigma(2)} + a_{\sigma(3)} &< a_{\sigma(1)} + a_4 + 3. \end{aligned} \quad (4.38)$$

Similarly,

$$\begin{aligned} \tilde{G}_4(y^{-1}) = & y^{-a_{\sigma(1)}-a_{\sigma(2)}-a_{\sigma(3)}+a_4} - y^{-a_{\sigma(1)}+a_{\sigma(2)}-a_{\sigma(3)}+a_4} \\ & - y^{-a_{\sigma(1)}-a_{\sigma(2)}+a_{\sigma(3)}+a_4}, \end{aligned} \quad (4.39)$$

with the counter terms

$$\tilde{H}_4(y) = \begin{cases} (a_{\sigma(1)} - a_{\sigma(2)} - a_{\sigma(3)} - a_4)y, & \text{if } \sum_i a_i \text{ is odd;} \\ \frac{1}{2}(a_{\sigma(1)} - a_{\sigma(2)} - a_{\sigma(3)} - a_4)y^2, & \text{if } \sum_i a_i \text{ is even.} \end{cases} \quad (4.40)$$

- Case 5: The twisted 4-gon condition is satisfied and in addition

$$\begin{aligned} a_{\sigma(1)} + a_{\sigma(2)} &\geq a_{\sigma(3)} + a_4 + 3 , \\ a_{\sigma(1)} + a_{\sigma(3)} &< a_{\sigma(2)} + a_4 + 3 , \\ a_{\sigma(2)} + a_{\sigma(3)} &< a_{\sigma(1)} + a_4 + 3 . \end{aligned} \tag{4.41}$$

In this case,

$$\tilde{G}_4(y^{-1}) = y^{-a_{\sigma(1)} - a_{\sigma(2)} - a_{\sigma(3)} + a_4} - y^{-a_{\sigma(1)} - a_{\sigma(2)} + a_{\sigma(3)} + a_4} , \tag{4.42}$$

with the counter terms

$$\tilde{H}_4(y) = \begin{cases} -2a_{\sigma(3)}y , & \text{if } \sum_i a_i \text{ is odd ;} \\ -a_{\sigma(3)}y^2 , & \text{if } \sum_i a_i \text{ is even .} \end{cases} \tag{4.43}$$

- Case 6: The twisted 4-gon condition is satisfied and in addition

$$\begin{aligned} a_1 + a_2 &< a_3 + a_4 + 3 , \\ a_1 + a_3 &< a_2 + a_4 + 3 , \\ a_2 + a_3 &< a_1 + a_4 + 3 . \end{aligned} \tag{4.44}$$

In this last case,

$$\tilde{G}_4(y^{-1}) = y^{-a_1 - a_2 - a_3 + a_4} , \tag{4.45}$$

and the counter terms are

$$\tilde{H}_4(y) = \begin{cases} (-a_1 - a_2 - a_3 + a_4)y , & \text{if } \sum_i a_i \text{ is odd ;} \\ \frac{1}{2}(-a_1 - a_2 - a_3 + a_4)y^2 , & \text{if } \sum_i a_i \text{ is even .} \end{cases} \tag{4.46}$$

5 Enumerating Cohomology

Having proved the first conjecture for all cyclic Abelian quivers, we now turn to the Higgs phase cohomology. Two aspects are of main concern in this section. The first is the proposed invariance of $H(M_k)/i_{M_k}^*(H(X_k))$. While more detailed information on the matter will be offered in the next section, including the general proof of the second conjecture at fully refined level and a routine for determining the full Hodge-decomposed cohomologies, we offer here a more preliminary, perhaps more intuitive view on the conjecture. The relatively simple ambient space X_k allows a pictorial realization of $i_{M_k}^*(H(X_k))$ via an n -dimensional triangular lattice, which produces a simple and intuitive proof of the invariance, albeit only at the level of

the numerical index. The second is an explicit and elementary counting formula for $D_k(-1) = \dim[i_{M_k}^*(H(X_k))]$, or equivalently the counting of the Coulomb phase ground states, which also follows from the same lattice realization of $i_{M_k}^*(H(X_k))$.

The usual numerical index for the Higgs phase is the Euler number $\chi(M_k)$, which can be computed as an integral of the top Chern class. Note that the total Chern class of M_k is given by

$$c(M_k) = \frac{\prod_{i \neq k} (1 + J_i)^{a_i}}{\left(1 + \sum_{i \neq k} J_i\right)^{a_k}}, \quad (5.1)$$

where J_i is the Kähler form of the \mathbb{CP}^{a_i-1} factor and the sum and the product over i run from 1 to $n+1$, except $i = k$. Then the evaluation of $\chi(M_k)$ follows

$$\chi(M_k) = \int_{X_k} \frac{\prod_{i \neq k} (1 + J_i)^{a_i}}{\left(1 + \sum_{i \neq k} J_i\right)^{a_k}} \cdot \left(\sum_{i \neq k} J_i\right)^{a_k}, \quad (5.2)$$

where we are supposed to extract the coefficient of $\prod_{i \neq k} J_i^{a_i-1}$ of the integrand. Alternatively and more explicitly, this Euler number can also be expressed as

$$\chi(M_k) = \prod_{i \neq k} a_i - \int_0^\infty ds \left(e^{-s} \prod_{i=1}^{n+1} L_{a_i-1}^1(s) \right) \quad (5.3)$$

with Laguerre polynomials $L_{a_i-1}^1(s)$.^{#7} This last expression (5.3) greatly simplifies the proof since the variation of the Euler number across walls, i.e. between adjacent branches, is only due to the first term. Thus k -independence of $\chi(M_k) - D_k(-1)$ is equivalent to the k -independence of

$$E_k \equiv \prod_{i \neq k} a_i - D_k(-1), \quad (5.4)$$

and hence, it is enough to show that $E_k = E_{k'}$ for any k, k' of a given cyclic quiver. We shall shortly reduce this to a counting problem in an n -dimensional triangular lattice.

Now, evaluation of $D_k(-1)$ can be conveniently cast into a counting of lattice points, thanks to Eq. (3.13).^{#8} With the Poincaré polynomial

$$\sum b_l(X_k) x^l = \frac{\prod_{i \neq k} (1 - x^{2a_i})}{(1 - x^2)^n} = \prod_{i \neq k} (1 + x^2 + x^4 + \cdots + x^{2(a_i-2)}), \quad (5.5)$$

^{#7}See Appendix E of Ref. [12] for $n = 2$ example. Generalization to $n > 2$ is straightforward.

^{#8}We are indebted to HwanChul Yoo for suggesting the lattice approach to the present counting problem, and also for suggesting the triangular lattice, as a slanted form of the rectangular lattice. The latter makes the invariance proof in subsequent subsections more intuitive and manifest.

we see that the Betti number $b_{2m}(X_k)$ of the ambient space equals the number of lattice points at level m , call it $\mathcal{N}_k(m)$, in an n -dimensional rectangular hyper-cube, bounded between 0 and $a_i - 1$ for each direction $i \neq k$. Later we will slant the lattice to make the bounded region into a hyper-parallelogram of lengths $a_i - 1$, instead of a hyper-cube. Here, however, let us simply imagine a rectangular lattice for the purpose of setting a combinatorial picture. The level of a lattice point $(\nu_1, \nu_2, \dots, \nu_n)$ is defined as $\sum \nu_j$, so the level spans from 0 to $\mu_k = \sum_{i \neq k} (a_i - 1) = \sum_{i \neq k} a_i - n$. Thus, we find that

$$D_k(-1) = \mathcal{N}_k(d_k/2) + 2 \sum_{m < d_k/2} \mathcal{N}_k(m) = \sum_{m \leq d_k/2} \mathcal{N}_k(m) + \sum_{m > \mu_k - d_k/2} \mathcal{N}_k(m) . \quad (5.6)$$

Since the total number of lattice points in such a hyper-cube is $\prod_{i \neq k} a_i$ by construction, we learn immediately that E_k is counted as the number of lattice points whose level m is between $d_k < m \leq \mu_k - d_k/2$

$$E_k = \prod_{i \neq k} a_i - D_k(-1) = \sum_{d_k/2 < m \leq \mu_k - d_k/2} \mathcal{N}_k(m) . \quad (5.7)$$

We shall prove the second conjecture by confirming that this counting is independent of k altogether.

A crucial ingredient for the proof comes from the fact that the upper bound for levels

$$\mu_k - d_k/2 = \sum_{i \neq k} a_i - n - \frac{\sum_{i \neq k} a_i - n - a_k}{2} = \frac{\sum_{i=1}^{n+1} a_i - n}{2} \quad (5.8)$$

is independent of k . As we shall see shortly, this allows a further transformation of the problem, once the lattice is slanted, to the counting of lattice points in an overlapping pair of mutually-inverted n -dimensional hyper-tetrahedrons (that is, n -simplices), of size $\sim (\sum_{i=1}^{n+1} (a_i - 1))/2$.

5.1 3-Gons

For a clear and simple picture, let us first focus on $n = 2$ cases. Once these are dealt with, generalization to higher n should be straightforward. Suppose that the linking numbers of a given quiver obey the inequalities

$$\begin{aligned} a_1 + 1 &\leq a_2 + a_3 , \\ a_2 + 1 &\leq a_3 + a_1 , \\ a_3 + 1 &\leq a_1 + a_2 , \end{aligned} \quad (5.9)$$

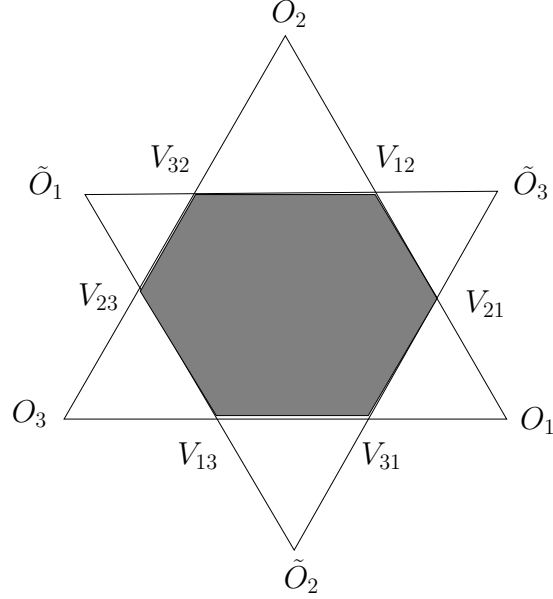


Figure 5.1: A pictorial representation of the Poincaré polynomials for a 3-gon quiver when $d_i \geq -1$ for $i = 1, 2, 3$. Two mutually-inverted equilateral triangles $O_1O_2O_3$ and $\tilde{O}_1\tilde{O}_2\tilde{O}_3$ are placed in a triangular lattice, overlapping with each other at a shaded hexagonal region. In each branch k , $E_k = \frac{a_1 a_2 a_3}{a_k} - D_k(-1)$ counts the number of lattice points inside this hexagon.

or equivalently,

$$d_i \geq -1, \quad i = 1, 2, 3, \quad (5.10)$$

and let us consider the “1-2 parallelogram” $O_3V_{31}\tilde{O}_3V_{32}$ in a triangular lattice (see Figure 5.1), whose two sides have a_1 and a_2 lattice points lying on them, respectively, *i.e.*, $\overline{O_3V_{31}} = a_1 - 1$ and $\overline{O_3V_{32}} = a_2 - 1$. It is easy to see that this 1-2 parallelogram naturally represents the Poincaré polynomial of the ambient space $X_3 = \mathbb{CP}^{a_1-1} \times \mathbb{CP}^{a_2-1}$ in branch 3. In particular, the fact that it contains total of $a_1 \cdot a_2$ lattice points inside implies that^{#9}

$$a_1 \cdot a_2 = \sum_{l=0}^{a_1+a_2-2} b_{2l}(X_3) = \sum_{l=0}^{2(a_1+a_2-2)} (-1)^l b_l(X_3) = \chi(X_3), \quad (5.11)$$

where, in the second step, vanishing of odd Betti numbers has been used.

Now, we place two parallel hyperplanes at lattice distances $\lceil \frac{d_3+1}{2} \rceil$ and $\lfloor \frac{d_3+1}{2} \rfloor$ (towards the inside region of the parallelogram), respectively, from the vertices O_3

^{#9}When counting lattice points inside a bounded region, we always include those lying on the boundary.

and \tilde{O}_3 , where $\lceil \cdot \rceil$ and $\lfloor \cdot \rfloor$ are the usual ceiling and floor functions:

$$\begin{aligned}\lceil x \rceil &= \min \{m \in \mathbb{Z} \mid m \geq x\}, \\ \lfloor x \rfloor &= \max \{m \in \mathbb{Z} \mid m \leq x\}.\end{aligned}$$

We also denote the eight lattice points by $\tilde{O}_2, V_{13}, V_{23}, \tilde{O}_1, O_1, V_{21}, V_{12}$ and O_2 , which arise from the intersection of these two hyperplanes with the four sides of the 1-2 parallelogram (the reason for this naming will become clearer when we consider general n cases). It is then straightforward to see that the quantity $E_3 = a_1 a_2 - D_3(-1)$ counts the lattice points inside the hexagonal region $V_{13}V_{31}V_{21}V_{12}V_{32}V_{23}$ (shaded in Figure 5.1) amongst the $a_1 a_2$ that are contained in the 1-2 parallelogram.

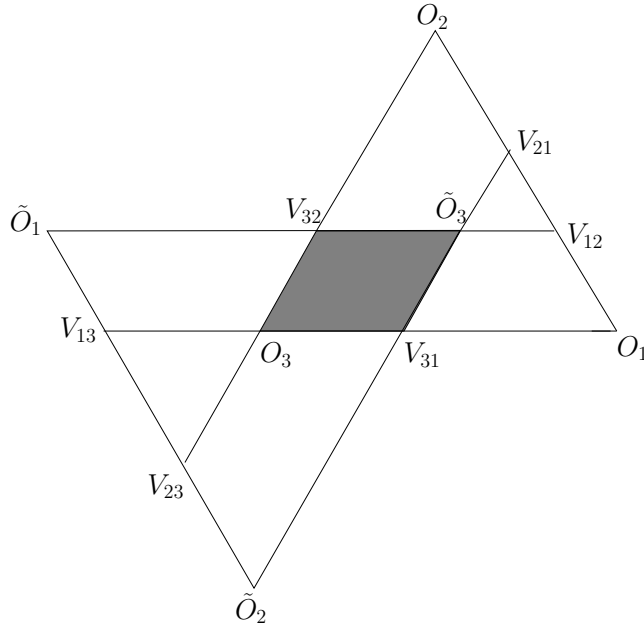


Figure 5.2: A pictorial representation of the Poincaré polynomials for a 3-gon quiver when $d_3 < -1$. Two mutually-inverted equilateral triangles $O_1O_2O_3$ and $\tilde{O}_1\tilde{O}_2\tilde{O}_3$ are placed in a triangular lattice, overlapping with each other at the shaded parallelogram. In each branch k , $E_k = \frac{a_1 a_2 a_3}{a_k} - D_k(-1)$ counts the number of lattice points inside this parallelogram. In particular, $\tilde{D}_3(-1) = 0$ and $E_3 = a_1 a_2$ in branch 3.

We shall now show that the quantity $E_1 = a_2 a_3 - D_1(-1)$ in branch 1 also counts the lattice points inside the same hexagon. This can most easily be understood by exchanging the roles of O_3 and O_1 . Note first that the sides of the equilateral triangle

$O_1O_2O_3$ have the following length:

$$\begin{aligned}
\overline{O_3O_1} &= \overline{O_3V_{31}} + \overline{V_{31}O_1} = \overline{O_3V_{31}} + \overline{V_{31}V_{21}} \\
&= (a_1 - 1) + \left((a_2 - 1) - \left\lfloor \frac{d_3 + 1}{2} \right\rfloor \right) \\
&= a_1 + a_2 - 2 - \left\lfloor \frac{a_1 + a_2 - a_3 - 1}{2} \right\rfloor \\
&= \left\lceil a_1 + a_2 - 2 - \frac{a_1 + a_2 - a_3 - 1}{2} \right\rceil \\
&= \left\lceil \frac{a_1 + a_2 + a_3 - 3}{2} \right\rceil \equiv l,
\end{aligned} \tag{5.12}$$

which is of a symmetric form under permutation of a_i 's. Next, let us also note that

$$\overline{O_1V_{12}} = \overline{O_3V_{32}} = a_2 - 1, \tag{5.13}$$

and that

$$\begin{aligned}
\overline{O_1V_{13}} &= l - \overline{O_3V_{13}} = l - \left\lfloor \frac{d_3 + 1}{2} \right\rfloor \\
&= \left\lceil \frac{a_1 + a_2 + a_3 - 3}{2} \right\rceil - \left\lfloor \frac{a_1 + a_2 - a_3 - 1}{2} \right\rfloor \\
&= \frac{a_1 + a_2 + a_3 - 3}{2} - \frac{a_1 + a_2 - a_3 - 1}{2} \\
&= a_3 - 1,
\end{aligned} \tag{5.14}$$

where the second last equality comes from the fact that $a_1 + a_2 + a_3 - 3$ and $a_1 + a_2 - a_3 - 1$ always have the same parity. So we conclude that the parallelogram $O_1V_{12}\tilde{O}_1V_{13}$ forms a “2-3 parallelogram” in that its two sides have a_2 and a_3 lattice points lying on them, respectively, *i.e.*, $\overline{O_1V_{12}} = a_2 - 1$ and $\overline{O_1V_{13}} = a_3 - 1$. What still remains to be verified for complete symmetry is that the hyperplane $\tilde{O}_2\tilde{O}_3$ is placed at lattice-distance $\lceil \frac{d_3+1}{2} \rceil$ from O_1 and the hyperplane O_2O_3 , at $\lfloor \frac{d_3+1}{2} \rfloor$ from \tilde{O}_1 . A few lines of algebra confirm these easily:

$$\overline{O_1V_{21}} = (a_2 - 1) - \left\lfloor \frac{d_3 + 1}{2} \right\rfloor = \left\lceil \frac{d_1 + 1}{2} \right\rceil, \tag{5.15}$$

$$\overline{\tilde{O}_1V_{23}} = \overline{V_{23}V_{32}} = (a_2 - 1) - \left\lfloor \frac{d_3 + 1}{2} \right\rfloor = \left\lceil \frac{d_1 + 1}{2} \right\rceil. \tag{5.16}$$

Therefore, E_1 also counts the lattice points in the same region $V_{13}V_{31}V_{21}V_{12}V_{32}V_{23}$, and hence, $E_3 = E_1$. A similar argument works in branch 2 and this completes the invariance proof

$$E_1 = E_2 = E_3, \tag{5.17}$$

in $n = 2$ cases.

Remarks:

- The hyperplane $\tilde{O}_1\tilde{O}_2$ at distance $\lceil \frac{d_3+1}{2} \rceil$ from O_3 is located closer to O_3 than the other hyperplane O_1O_2 is. This is guaranteed by

$$\left\lceil \frac{d_3 + 1}{2} \right\rceil = \left\lceil \frac{a_1 + a_2 - a_3 - 1}{2} \right\rceil \leq \left\lceil \frac{a_1 + a_2 + a_3 - 3}{2} \right\rceil = l.$$

Thus, the lattice points in the hexagonal region indeed have intermediate levels as described in Eq. (5.7).

- We have assumed from the start that the linking numbers a_i obeyed the inequalities (5.9). However, our argument works in general if we consider all the edge lengths as signed ones. Note that only one of the three inequalities (5.9) can be violated at a time, and suppose $d_3 < -1$. Then having the length $\lceil \frac{d_3+1}{2} \rceil$ negative, for example, means that the hyperplane $\tilde{O}_1\tilde{O}_2$ is located outside the 1-2 parallelogram $O_3V_{31}\tilde{O}_3V_{32}$ (see Figure 5.2). It is easy to see in this case that \tilde{O}_3 sits inside the triangle $O_1O_2O_3$. So the shaded region, which is the overlap of the two triangles $O_1O_2O_3$ and $\tilde{O}_1\tilde{O}_2\tilde{O}_3$, is the entire 1-2 parallelogram and hence, $E_3 = a_1a_2$, or equivalently, $D_3(-1) = 0$.

5.2 $(n + 1)$ -Gons

Having had enough experiences with $n = 2$ cases, we can rather formally proceed to general n cases. Let $S_n = O_1O_2 \cdots O_nO_{n+1}$ be the n -simplex of edge length

$$l_n \equiv \left\lceil \frac{\sum_{i=1}^{n+1} a_i - (n+1)}{2} \right\rceil,$$

where a_i are the $n + 1$ linking numbers of a given $(n + 1)$ -gon quiver. Let us now place S_n in an n -dimensional triangular lattice so that O_{n+1} sits at the origin and the edges $\overline{O_{n+1}O_i}$ lie along the positive x_i -axes for $i = 1, 2, \dots, n$ (see Figure 5.3 for an $n = 3$ case, in which we have a 3-simplex, namely, a tetrahedron). For simplicity, let us suppose that the linking numbers obey the inequalities

$$\begin{aligned} a_1 + (n-1) &\leq a_2 + a_3 + \cdots + a_{n+1}, \\ a_2 + (n-1) &\leq a_1 + a_3 + \cdots + a_{n+1}, \\ &\vdots \\ a_{n+1} + (n-1) &\leq a_1 + a_2 + \cdots + a_n, \end{aligned} \tag{5.18}$$

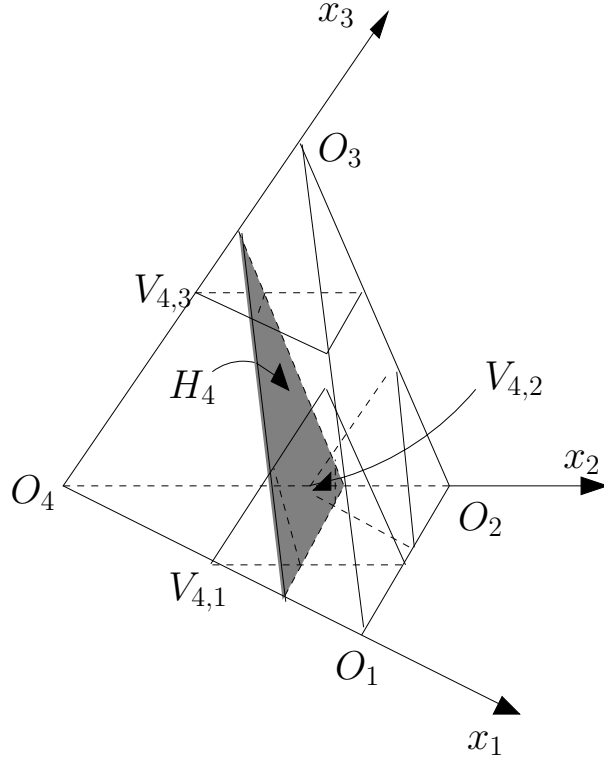


Figure 5.3: A pictorial representation of the Poincaré polynomials for a 4-gon quiver when $d_i \geq -1$ for $i = 1, 2, 3, 4$.

or equivalently,

$$d_i \geq -1, \text{ for } i = 1, 2, \dots, n+1. \quad (5.19)$$

Now, we place $n+1$ hyperplanes \mathcal{H}_j at lattice distances $\left\lceil \frac{d_j+1}{2} \right\rceil$ (towards the inside region of S_n) from the vertices O_j . These hyperplanes \mathcal{H}_j form another simplex \tilde{S}_n , which has an inverted shape compared with the original simplex S_n . Let $I \equiv S_n \cap \tilde{S}_n$ be the overlap of S_n with \tilde{S}_n . We claim that the number of lattice points in I equals E_k in any branches, which implies that E_k is independent of the branch choice, k .

Thanks to the symmetry, without loss of generality, we may only consider branch $n+1$. Let us denote by $V_{n+1,j}$ the x_j -cuts of the hyperplanes \mathcal{H}_j for $j = 1, \dots, n$. It is easy to see that these n hyperplanes $\mathcal{H}_1, \dots, \mathcal{H}_n$ together with the n coordinate planes, $x_j = 0$ for $j = 1, \dots, n$, form a bounded hyper-parallelogram (see Figure 5.4 for an $n = 3$ example). The n sides of this object have the following lengths

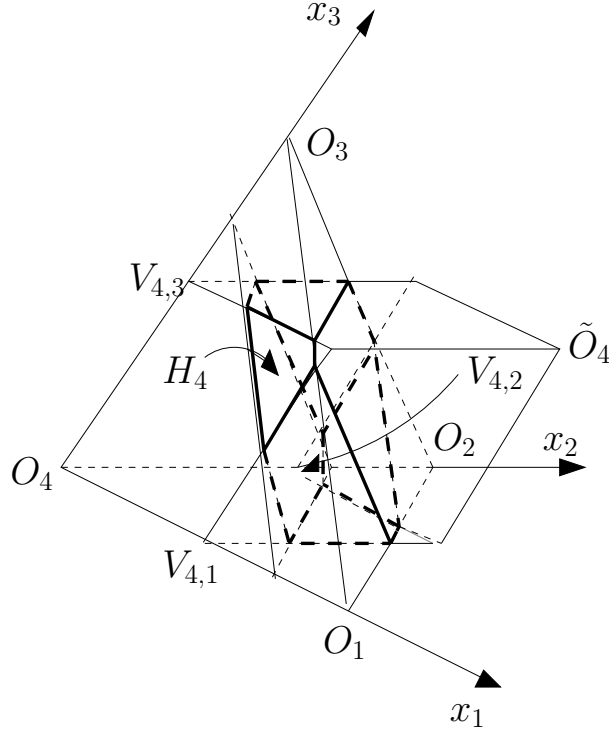


Figure 5.4: A pictorial representation of the Poincaré polynomials for a 4-gon quiver when $d_i \geq -1$ for $i = 1, 2, 3, 4$. A hyper-parallelotope with sides of lengths $a_i - 1$ has been added to Figure 5.3. Counting of the lattice points inside the region with thick edges gives $E_k = \frac{a_1 a_2 a_3}{a_k} - D_k(-1)$ in any branches k .

$$\begin{aligned}
\overline{O_{n+1}V_{n+1,j}} &= l_n - \left\lceil \frac{d_j + 1}{2} \right\rceil \\
&= \left\lceil \frac{\sum_{i=1}^{n+1} a_i - (n+1)}{2} \right\rceil - \left\lceil \frac{\sum_{i=1}^{n+1} a_i - 2a_j - n + 1}{2} \right\rceil \\
&= a_j - 1,
\end{aligned} \tag{5.20}$$

where in the last step we have used the fact that $\sum_{i=1}^{n+1} a_i - (n+1)$ and $\sum_{i=1}^{n+1} a_i - 2a_j - n + 1$ have the same parity. Therefore, this hyper-parallelotope naturally represents the Poincaré polynomial of the ambient space $X_{n+1} = \prod_{i=1}^n \mathbb{CP}^{a_i-1}$ in branch $n+1$.

Let \tilde{O}_{n+1} be the opposite vertex of O_{n+1} in this hyper-parallelotope. According to Eq. (5.20), this vertex \tilde{O}_{n+1} has the coordinates $(a_1 - 1, \dots, a_n - 1)$. Let us now note that by construction the hyperplane \mathcal{H}_{n+1} is located at lattice-distance $\left\lceil \frac{d_{n+1}+1}{2} \right\rceil$ from O_{n+1} , and also that the face $O_1 O_2 \cdots O_n$ cuts the hyper-parallelotope at the

following distance from \tilde{O}_{n+1} :

$$\left(\sum_{i=1}^n a_i - n \right) - l_n = \sum_{i=1}^n a_i - n - \left\lceil \frac{\sum_{i=1}^{n+1} a_i - (n+1)}{2} \right\rceil = \left\lfloor \frac{d_{n+1} + 1}{2} \right\rfloor. \quad (5.21)$$

Therefore, as in the 2-simplex cases of the previous subsection, we see that $E_{n+1} = \prod_{i=1}^n a_i - D_{n+1}(-1)$ counts the lattice points in region I .

We performed the above computation for the branch $n+1$, which pictorially corresponds to picking out one particular vertex O_{n+1} in the simplex $O_1 O_2 \cdots O_{n+1}$. However, there is nothing special about O_{n+1} among the $n+1$ vertices, and we could have done exactly the same for any O_k ; The final expression is always the count of the lattice points in $I = S_n \cap \tilde{S}_n$. This proves the invariance of E_k .

Remarks:

- The intersection region $I = S_n \cap \tilde{S}_n$ always exists. To see this, we may only check in branch $n+1$ that the hyperplane \mathcal{H}_{n+1} is located closer to O_{n+1} than the face $O_1 \cdots O_n$ is, *i.e.*, the lattice points on \mathcal{H}_{n+1} have a lower level than those on the face $O_1 \cdots O_n$. One can easily check this:

$$\left\lfloor \frac{d_{n+1} + 1}{2} \right\rfloor = \left\lfloor \frac{\sum_{i=1}^{n+1} a_i - 2a_{n+1} - n + 1}{2} \right\rfloor \leq \left\lfloor \frac{\sum_{i=1}^{n+1} a_i - (n+1)}{2} \right\rfloor = l_n.$$

- We have assumed that the linking numbers a_i obeyed the inequalities (5.18). However, our argument also works if one of the $n+1$ inequalities is violated (only one can be violated at a time as in $n=2$ cases). Suppose that the violation was by d_{n+1} , that is, $d_{n+1} < -1$. Having the length $\left\lfloor \frac{d_{n+1} + 1}{2} \right\rfloor$ negative means that the hyperplane \mathcal{H}_{n+1} is located outside the hyper-parallellogram. It is easy to see in this case that \tilde{O}_{n+1} sits inside the simplex S . So the intersecting region I is the entire hyper-parallellogram and hence, $E_{n+1} = \prod_{i=1}^n a_i$, or equivalently, $D_{n+1}(-1) = 0$.

5.3 Formula for $D_k(-1) = \dim i_{M_k}^*(H(X_k))$

Before closing the section, let us now enumerate the states explicitly. As we already have a simple routine for generating Euler numbers, here we give a similarly simple

routine for computing $D_k(-1)$. Taking the difference between the two in any one of the branches would then give the total number of Intrinsic Higgs states. In the next section, we will find yet another method which does not only enumerate states but also catalog (Intrinsic) Higgs phase states according to global charges.

Denoting the invariant (5.4) of the quiver by $E \equiv E_1 = E_2 = \dots = E_{n+1}$, we can write

$$D_k(-1) = \prod_{i \neq k} a_i - E . \quad (5.22)$$

According to the first conjecture, these numbers should equal the Coulomb phase indices up to a sign. While we investigate the refined version of these quantities in the next section, we also record here a general counting formula and its very simple geometric interpretation in terms of the triangular lattice and hyper-parallelogram introduced in section 5.2. It turns out that

$$D_k(-1) = \prod_{i \neq k} a_i - V_n(l_n) + \sum_{r=1}^{n+1} (-1)^{r-1} \sum_{i_1 < \dots < i_r} V_n(l_n - \sum_{\alpha=1}^r a_{i_\alpha}) , \quad (5.23)$$

where $V_n(m)$, for $m \geq 0$, is the number of lattice points inside the n -simplex with edge length m , and is defined to be 1 and 0, respectively, for $m = 0$ and $m < 0$.

Let us present a sketchy derivation of the above formula. The idea is first to find a recursive expression for the invariant quantity E , which counts the lattice points in the overlap region I of the two n -simplices, $S_n = O_1 \dots O_{n+1}$ and $\tilde{S}_n = \tilde{O}_1 \dots \tilde{O}_{n+1}$, as defined in section 5.2. For this, one starts from the simplex S_n , in which there are total of $V_n(l_n)$ lattice points. Amongst them, those lattice points inside the smaller n -simplex with $n+1$ vertices O_k , $V_{i,k}(i \neq k)$ must be excluded for each k , while those on its face opposite to O_k must still be counted. After shifting this face by one lattice unit towards O_k , the size of this k -th simplex becomes

$$l_n - (a_k - 1) - 1 = l_n - a_k , \quad (5.24)$$

and hence, we have to subtract $V_n(l_n - a_k)$ from $V_n(l_n)$, for each k . This, however, is not quite the final answer since a pair from these $n+1$ simplices, say i -th and j -th ones, may also overlap with each other at a smaller simplex with size $l_n - a_i - a_j + 2$ (see, for instance, Figure 5.3 or 5.4). Again, one should be careful about the shift of faces for correct counting, and in this case, it turns out that the shift by a unit has to be made twice. Thus, we add $V_n(l_n - a_i - a_j)$ back in, for each pair (i, j) with $l_n - a_i - a_j$ non-negative, etc., and in the end obtain the expression (5.23) for $D_k(-1)$.

Note that the discrete volume, $V_n(m)$, can be computed inductively by using the recursion relation

$$V_n(m) = V_n(m-1) + V_{n-1}(m) , \quad (5.25)$$

with the initial condition

$$V_1(m) = m + 1 , \quad (5.26)$$

which merely counts the lattice points on a line segment of length m . This leads to the simple expression

$$V_n(m) = \binom{n+m}{n} , \quad (5.27)$$

for non-negative m and zero otherwise. Eq. (5.23) for $D_k(-1)$ can then be re-written as the alternating sum of binomial coefficients,

$$\begin{aligned} D_k(-1) = & \prod_{i \neq k} a_i - \binom{n+l_n}{n} + \sum_{i_1} \binom{n+l_n-a_{i_1}}{n} \\ & - \sum_{i_1 < i_2} \binom{n+l_n-a_{i_1}-a_{i_2}}{n} + \cdots , \end{aligned} \quad (5.28)$$

where the ellipsis means that we keep adding or subtracting the binomial coefficients for all collections of subscripts, i 's, as long as $m = l_n - a_{i_1} - a_{i_2} - \cdots$ is non-negative. One should take care to truncate this series at places where the “length” m turns negative, since the binomial function can give a (unwanted) nonzero number when the upper argument $n+m$ is a negative integer.

6 Refined Index $\Omega_{\text{Higgs}}^{(k)}(y)$ and the Second Conjecture

In this section, we finally come to the general proof of the second conjecture at fully refined level. In the previous section, we already observed that the degeneracy of the Intrinsic Higgs sector, or equivalently the difference between the Higgs phase degeneracy and the Coulomb phase degeneracy, is independent of the branch choice, k . The aim here is to elevate that observation to fully refined level with all angular momentum and $U(1)'_R$ charges manifest. In addition, we will display many examples (in subsection 6.2) which will illuminate the Higgs phase ground state sector numerically.

Let us recall how the protected spin character reduces in the Higgs phase to

$$\Omega_{\text{Higgs}}^{(k)}(y) = \text{tr} (-1)^{2L_3} y^{2L_3+2\mathcal{I}} = \text{tr}_{H(M_k)} (-1)^{p+q-d_k} y^{2p-d_k} . \quad (6.1)$$

Its analog for the Intrinsic Higgs sector,

$$\Omega_{\text{Higgs}}^{(k)}(y) \Big|_{\text{Intrinsic}} = \text{tr}_{H(M_k)/i_{M_k}^*(H(X_k))} (-1)^{p+q-d_k} y^{2p-d_k} , \quad (6.2)$$

should be a refined invariant insensitive to wall-crossings.

For cyclic quivers, luckily, these refined indices are also relatively easy to compute. For this, recall the notion of refined Euler numbers on a Kähler manifold M ,

$$\chi^p(M) = \sum_{q \geq 0} (-1)^q h^{p,q}(M), \quad (6.3)$$

with the usual Hodge numbers $h^{p,q} = \dim H^{p,q}(M)$, and their generating function^{#10}

$$\chi_\xi(M) = \sum_{p \geq 0} \chi^p(M) \xi^p, \quad (6.4)$$

which we call the refined Euler character.

Taking $M = M_k$ and comparing $\chi_\xi(M_k)$ against $\Omega_{\text{Higgs}}^{(k)}(y)$ above, we find the two are related straightforwardly as

$$\Omega_{\text{Higgs}}^{(k)}(y) = (-y)^{-d_k} \chi_{\xi=-y^2}(M_k). \quad (6.5)$$

This identification can be understood from discussions of section 2, where we have identified the supercharges that commute with $y^{2L_3+2\mathcal{I}}$ to be $\bar{\partial}$ and $\bar{\partial}^\dagger$. On the other hand, $\bar{\partial} + \bar{\partial}^\dagger$ is nothing but the elliptic operator responsible for the complex $\bigoplus_q H^{p,q}(M_k)$; the numerical index for this complex, for each $0 \leq p \leq d_k$, is precisely $\chi^p(M_k)$ in (6.3).

While this expression is only an index, it actually carries the full cohomology information of M_k ; For any given p , only two entries in the Hodge diamond contribute to $\chi_\xi(M_k)$, namely $h^{p,p}$ and h^{p,d_k-p} . The former belongs to $i_{M_k}^*(H(X_k))$ while the latter belongs to the Intrinsic Higgs sector; the only exception to this occurs for $p = d_k/2$ with d_k even, in which case the two sector gets mixed in the single entry $h^{d_k/2, d_k/2}$. On the other hand, $i_{M_k}^*(H(X_k))$ is entirely captured by the reduced Poincaré polynomial, $D_k(x)$, which together with $\chi_\xi(M_k)$ determines $H^{p,q}(M_k)$ entirely. In section 6.2, we will present explicit examples of these refined indices as well as the Hodge diamonds.

Restricting the trace to $i_{M_k}^*(H(X_k))$, we similarly have

$$\Omega_{\text{Higgs}}^{(k)}(y) \Big|_{i_{M_k}^*(H(X_k))} = (-y)^{-d_k} \chi_{\xi=-y^2}(M_k) \Big|_{i_{M_k}^*(H(X_k))} = (-y)^{d_k} D_k(-y), \quad (6.6)$$

^{#10}The usual Euler character is obtained by substituting $\xi = -1$:

$$\chi_\xi(M) \Big|_{\xi=-1} = \sum_{p,q \geq 0} (-1)^{p+q} h^{p,q}(M) = \chi(M).$$

and for the intrinsic sector,

$$\begin{aligned} \Omega_{\text{Higgs}}^{(k)}(y) \Big|_{\text{Intrinsic}} &= \Omega_{\text{Higgs}}^{(k)}(y) - \Omega_{\text{Higgs}}^{(k)}(y) \Big|_{i_{M_k}^*(H(X_k))} \\ &= (-y)^{-d_k} \chi_{\xi=-y^2}(M_k) - (-y)^{-d_k} D_k(-y) . \end{aligned} \quad (6.7)$$

The second conjecture says that this last expression is an invariant of quiver. In subsection 6.3, we will give a complete and rigorous proof of this assertion by demonstrating its k -independence.

6.1 Computing the Refined Euler Character

For cyclic Abelian quivers, we already have a general formula for $D_k(x)$, so it is now a matter of finding $\chi_\xi(M_k)$. We shall use the Hirzebruch-Riemann-Roch formula [24] to find that $\chi_\xi(M_k)$ have the following integral representation,

$$\chi_\xi(M_k) = \frac{1}{(1+\xi)^n} \int_{X_k} \left[\prod_{i \neq k} \left(J_i \frac{1+\xi e^{-J_i}}{1-e^{-J_i}} \right)^{a_i} \right] \cdot \left(\frac{1-e^{-\sum_{i \neq k} J_i}}{1+\xi e^{-\sum_{i \neq k} J_i}} \right)^{a_k} , \quad (6.8)$$

which is essentially given as the coefficient of $\prod_{i \neq k} J_i^{a_i-1}$ in the Taylor expansion of the integrand. As before, J_i are the Kähler forms of CP^{a_i-1} in X_k .

To derive the above formula, let us start from recalling the definitions of relevant topological quantities and their basic properties. For a vector bundle E on X , we set

$$H^{p,q}(X, E) := H^q(X, \Omega^p(E)) , \quad h^{p,q}(X, E) := \dim H^{p,q}(X, E) , \quad (6.9)$$

and define the holomorphic Euler character

$$\chi^p(X, E) := \sum_{q \geq 0} (-1)^q h^{p,q}(X, E) , \quad (6.10)$$

whose generating function is denoted by

$$\chi_\xi(X, E) := \sum_{p \geq 0} \chi^p(X, E) \xi^p . \quad (6.11)$$

We write $\chi_\xi(X)$ if E is the trivial holomorphic line bundle, reducing to Eq. (6.4). Let us also define

$$\text{ch}_\xi(E) := \sum_{p \geq 0} \text{ch}(\wedge^p E) \xi^p . \quad (6.12)$$

Then, for a line bundle L with $c_1(L) = t \in H^2(X, \mathbb{Z})$, we have

$$\text{ch}_\xi(L) = 1 + \xi e^t, \quad (6.13)$$

$$\text{td}(L) = \frac{t}{1 - e^{-t}}, \quad (6.14)$$

where $\text{td}(L)$ is the Todd genus of L . We finally note that, if the bundle E is given by the extension sequence

$$0 \rightarrow E_1 \rightarrow E \rightarrow E_2 \rightarrow 0, \quad (6.15)$$

then the Todd genus and the Chern character have the following multiplicative properties:

$$\text{td}(E) = \text{td}(E_1) \cdot \text{td}(E_2), \quad (6.16)$$

$$\text{ch}_\xi(E) = \text{ch}_\xi(E_1) \cdot \text{ch}_\xi(E_2). \quad (6.17)$$

For a single projective space $X = \mathbb{CP}^{a-1}$, the Euler sequence

$$0 \rightarrow \mathcal{O}_X \rightarrow \mathcal{O}_X(1)^{\oplus a} \rightarrow \mathcal{T}X \rightarrow 0, \quad (6.18)$$

leads to

$$\text{td}(\mathcal{T}X) = \left(\frac{J}{1 - e^{-J}} \right)^a, \quad (6.19)$$

$$\text{ch}_\xi(\mathcal{T}^*X) = \frac{(1 + \xi e^{-J})^a}{1 + \xi}, \quad (6.20)$$

where J is the Kähler form of X . Due to the multiplicative property of Todd genus and Chern character, they have a natural generalization to the product of projective spaces $X_k = \prod_{i \neq k} \mathbb{CP}^{a_i-1}$, which are the ambient varieties of our concern.

We are now ready to address the χ_ξ -character of the complete intersection M_k . The Adjunction formula dictates that we have the following relations

$$\text{td}(\mathcal{T}M_k) = \left[\prod_{i \neq k} \left(\frac{J_i}{1 - e^{-J_i}} \right)^{a_i} \right] \cdot \left(\frac{1 - e^{-\sum_{i \neq k} J_i}}{\sum_{i \neq k} J_i} \right)^{a_k}, \quad (6.21)$$

$$\text{ch}_\xi(\mathcal{T}^*M_k) = \left[\prod_{i \neq k} \frac{(1 + \xi e^{-J_i})^{a_i}}{1 + \xi} \right] \cdot \left(\frac{1}{1 + \xi e^{-\sum_{i \neq k} J_i}} \right)^{a_k}, \quad (6.22)$$

where J_i is the Kähler form from each \mathbb{CP}^{a_i-1} factor in X_k . Therefore, upon applying the Hirzebruch-Riemann-Roch formula [24], we have

$$\begin{aligned}
\chi_\xi(M_k) &= \int_{M_k} \text{td}(\mathcal{T}M_k) \cdot \text{ch}_\xi(\mathcal{T}^*M_k) \\
&= \int_{X_k} \text{td}(\mathcal{T}M_k) \cdot \text{ch}_\xi(\mathcal{T}^*M_k) \cdot \left(\sum_{i \neq k} J_i \right)^{a_k} \\
&= \frac{1}{(1+\xi)^n} \int_{X_k} \left[\prod_{i \neq k} \left(J_i \frac{1+\xi e^{-J_i}}{1-e^{-J_i}} \right)^{a_i} \right] \cdot \left(\frac{1-e^{-\sum_{i \neq k} J_i}}{1+\xi e^{-\sum_{i \neq k} J_i}} \right)^{a_k}, \quad (6.23)
\end{aligned}$$

and hence, arrive at the expression (6.8) as promised. Note that in the second step the Poincaré dual of $[M_k]$ was multiplied to elevate the integral to the ambient space X_k , and in the third step Eqs. (6.21) and (6.22) have been used.

6.2 Indices and Hodge Numbers : Numerical Illustrations

Let us study the refined indices and the resulting Hodge numbers for specific examples. To understand typical shape of Hodge diamonds, let us first take a 3-gon example with $a_1 = a_2 = a_3 = 8$, so that all three branches are identical with the complex dimension $d = 6$. In this case, the Hodge diamond of $h^{p,q}$'s can be drawn as

$$\begin{array}{cccccccc}
& & & & 1 & & & \\
& & & & 0 & & 0 & \\
& & & 0 & 2 & & 0 & \\
& & 0 & 0 & 0 & & 0 & \\
& 0 & 0 & 0 & 3 & & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
& 0 & 322 & 6803 & 18216 & 6803 & 322 & 1 \\
& 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
& & 0 & 0 & 3 & & 0 & 0 \\
& & 0 & 0 & 0 & & 0 & \\
& & & 0 & 2 & & 0 & \\
& & & 0 & 0 & & 0 & \\
& & & & 1 & & &
\end{array}$$

The horizontal line represents (mostly) intrinsic Higgs states, except four out of the $h^{3,3} = 18216 = 18212 + 4$ states, that belong to $i_M^*(H(X))$. This illustrates how the Intrinsic Higgs states and the rest separate neatly (except at $h^{d/2,d/2}$ when d is even) into middle horizontal and middle vertical part of the Hodge diamond. This cross-like pattern with horizontal Intrinsic part and vertical pulled-back part, with

possible overlap at the center $h^{d/2,d/2}$ when d is even, is a general feature of cyclic Abelian quivers.

To illustrate what happens when different branches are topologically distinct, we take another 3-gon example, with $(a_1, a_2, a_3) = (4, 5, 6)$, for which we obtain the following Hodge diamonds in branches 1, 2 and 3, respectively:

$$\begin{array}{ccccccccc}
 & & & & 1 & & & & \\
 & & & & 0 & & 0 & & \\
 & & & 0 & & 2 & & 0 & \\
 & & 0 & & 0 & & 0 & & 0 \\
 & 0 & & 0 & & 3 & & 0 & & 0 \\
 0 & & 0 & & 26 & & 26 & & 0 & & 0 \ , \quad 0 & & 0 & & 26 & & 26 & & 0 \ , \quad 26 & & 26 \ . \\
 & & 0 & & 0 & & 3 & & 0 & & 0 & & 0 & & 2 & & 0 & & 1 \\
 & & 0 & & 0 & & 0 & & 0 & & 0 & & 0 & & 0 & & 0 & & 1 \\
 & & 0 & & 2 & & 0 & & 0 & & 1 & & & & & & & & \\
 & & 0 & & 0 & & & & & & & & & & & & & \\
 & & & & 1 & & & & & & & & & & & & &
 \end{array}$$

Note how the same middle line is repeated. In this case, the complex dimensions of M_k 's are odd, so Coulomb states do not mix in the middle cohomology, which then represents the Intrinsic Higgs states entirely. Equivalently, these data are encoded in the refined index of the Higgs phase. For example, in branch 1, the index is given by

$$\Omega_{\text{Higgs}}^{(1)}(y) = -\frac{1}{y^5} - \frac{2}{y^3} + \frac{23}{y} + 23y - 2y^3 - y^5 \ , \quad (6.24)$$

and when restricted to the pulled-back cohomology, by

$$\Omega_{\text{Higgs}}^{(1)}(y) \Big|_{i_{M_1}^*(H(X_1))} = (-y)^{-d_1} D_1(-y) = -\frac{1}{y^5} - \frac{2}{y^3} - \frac{3}{y} - 3y - 2y^3 - y^5 \ . \quad (6.25)$$

Then the Intrinsic Higgs sector has the following refined index

$$\begin{aligned}
 \Omega_{\text{Higgs}}^{(1)}(y) \Big|_{\text{Intrinsic}}^{(4,5,6)} &= \Omega_{\text{Higgs}}^{(1)}(y) - \Omega_{\text{Higgs}}^{(1)}(y) \Big|_{i_{M_1}^*(H(X_1))} \\
 &= \frac{26}{y} + 26y \ , \quad (6.26)
 \end{aligned}$$

which, in this case, describes the invariant middle cohomology in full detail. Repeating the same exercise for branches 2 and 3, we find

$$\Omega_{\text{Higgs}}^{(1)}(y) \Big|_{\text{Intrinsic}}^{(4,5,6)} = \Omega_{\text{Higgs}}^{(2)}(y) \Big|_{\text{Intrinsic}}^{(4,5,6)} = \Omega_{\text{Higgs}}^{(3)}(y) \Big|_{\text{Intrinsic}}^{(4,5,6)} = \frac{26}{y} + 26y \ , \quad (6.27)$$

as advertised. Finally, we record a few more, less-trivial, examples. The first is another 3-gon example with $(a_1, a_2, a_3) = (15, 16, 17)$. The total refined Higgs phase indices in the three branches are given as,

$$\begin{aligned}
\Omega_{\text{Higgs}}^{(1)}(y) &= 1/y^{16} + 2/y^{14} + 1668/y^{12} + 724678/y^{10} + 60686568/y^8 \\
&\quad + 1523273850/y^6 + 13886938956/y^4 + 50685934046/y^2 + 77668453896 \\
&\quad + 50685934046y^2 + 13886938956y^4 + 1523273850y^6 \\
&\quad + 60686568y^8 + 724678y^{10} + 1668y^{12} + 2y^{14} + y^{16} , \\
\Omega_{\text{Higgs}}^{(2)}(y) &= 1/y^{14} + 1667/y^{12} + 724677/y^{10} + 60686567/y^8 \\
&\quad + 1523273849/y^6 + 13886938955/y^4 + 50685934045/y^2 + 77668453895 \\
&\quad + 50685934045y^2 + 13886938955y^4 + 1523273849y^6 \\
&\quad + 60686567y^8 + 724677y^{10} + 1667y^{12} + y^{14} , \\
\Omega_{\text{Higgs}}^{(3)}(y) &= 1666/y^{12} + 724676/y^{10} + 60686566/y^8 + 1523273848/y^6 , \\
&\quad + 13886938954/y^4 + 50685934044/y^2 + 77668453894 \\
&\quad + 50685934044y^2 + 13886938954y^4 + 1523273848y^6 \\
&\quad + 60686566y^8 + 724676y^{10} + 1666y^{12} .
\end{aligned} \tag{6.28}$$

Similarly, refined Higgs indices for the pulled-back part are,

$$\begin{aligned}
(-y)^{-d_1} D_1(-y) &= 1/y^{16} + 2/y^{14} + 3/y^{12} + 4/y^{10} + 5/y^8 + 6/y^6 + 7/y^4 + 8/y^2 \\
&\quad + 9 + 8y^2 + 7y^4 + 6y^6 + 5y^8 + 4y^{10} + 3y^{12} + 2y^{14} + y^{16} , \\
(-y)^{-d_2} D_2(-y) &= 1/y^{14} + 2/y^{12} + 3/y^{10} + 4/y^8 + 5/y^6 + 6/y^4 + 7/y^2 \\
&\quad + 8 + 7y^2 + 6y^4 + 5y^6 + 4y^8 + 3y^{10} + 2y^{12} + y^{14} , \\
(-y)^{-d_3} D_3(-y) &= 1/y^{12} + 2/y^{10} + 3/y^8 + 4/y^6 + 5/y^4 + 6/y^2 \\
&\quad + 7 + 6y^2 + 5y^4 + 4y^6 + 3y^8 + 2y^{10} + y^{12} ,
\end{aligned} \tag{6.29}$$

from which we find

$$\begin{aligned}
\Omega_{\text{Higgs}}(y) \Big|_{\text{Intrinsic}}^{(15,16,17)} &= \Omega_{\text{Higgs}}^{(k)}(y) - (-y)^{-d_k} D_k(-y) \\
&= 1665/y^{12} + 724674/y^{10} + 60686563/y^8 + 1523273844/y^6 \\
&\quad + 13886938949/y^4 + 50685934038/y^2 + 77668453887 \\
&\quad + 50685934038y^2 + 13886938949y^4 + 1523273844y^6 \\
&\quad + 60686563y^8 + 724674y^{10} + 1665y^{12} ,
\end{aligned} \tag{6.30}$$

independent of k , again as advertised. The second example is a 4-gon case with $(a_1, a_2, a_3, a_4) = (5, 6, 7, 8)$. The total refined Higgs phase indices in the four branch

are give as,

$$\begin{aligned}
\Omega_{\text{Higgs}}^{(1)}(y) &= -1/y^{13} - 3/y^{11} - 6/y^9 + 4415/y^7 + 362210/y^5 + 5127653/y^3 + 18229383/y \\
&\quad + 18229383y + 5127653y^3 + 362210y^5 + 4415y^7 - 6y^9 - 3y^{11} - y^{13} , \\
\Omega_{\text{Higgs}}^{(2)}(y) &= -1/y^{11} - 3/y^9 + 4419/y^7 + 362215/y^5 + 5127659/y^3 + 18229390/y \\
&\quad + 18229390y + 5127659y^3 + 362215y^5 + 4419y^7 - 3y^9 - y^{11} , \\
\Omega_{\text{Higgs}}^{(3)}(y) &= -1/y^9 + 4422/y^7 + 362219/y^5 + 5127664/y^3 + 18229395/y \\
&\quad + 18229395y + 5127664y^3 + 362219y^5 + 4422y^7 - y^9 , \\
\Omega_{\text{Higgs}}^{(4)}(y) &= 4424/y^7 + 362222/y^5 + 5127668/y^3 + 18229400/y \\
&\quad + 18229400y + 5127668y^3 + 362222y^5 + 4424y^7 ,
\end{aligned} \tag{6.31}$$

while refined indices for the pulled-back part are,

$$\begin{aligned}
(-y)^{-d_1} D_1(-y) &= -1/y^{13} - 3/y^{11} - 6/y^9 - 10/y^7 - 15/y^5 - 21/y^3 - 27/y \\
&\quad - 27y - 21y^3 - 15y^5 - 10y^7 - 6y^9 - 3y^{11} - y^{13} , \\
(-y)^{-d_2} D_2(-y) &= -1/y^{11} - 3/y^9 - 6/y^7 - 10/y^5 - 15/y^3 - 20/y \\
&\quad - 20y - 15y^3 - 10y^5 - 6y^7 - 3y^9 - y^{11} , \\
(-y)^{-d_3} D_3(-y) &= -1/y^9 - 3/y^7 - 6/y^5 - 10/y^3 - 15/y \\
&\quad - 15y - 10y^3 - 6y^5 - 3y^7 - y^9 , \\
(-y)^{-d_4} D_4(-y) &= -1/y^7 - 3/y^5 - 6/y^3 - 10/y \\
&\quad - 10y - 6y^3 - 3y^5 - y^7 ,
\end{aligned} \tag{6.32}$$

which, together, gives

$$\begin{aligned}
\Omega_{\text{Higgs}}(y) \Big|_{\text{Intrinsic}}^{(5,6,7,8)} &= \Omega_{\text{Higgs}}^{(k)}(y) - (-y)^{-d_k} D_k(-y) \\
&= 4425/y^7 + 362225/y^5 + 5127674/y^3 + 18229410/y \\
&\quad + 18229410y + 5127674y^3 + 362225y^5 + 4425y^7 ,
\end{aligned} \tag{6.33}$$

independent of k . The promised invariance can be seen very clearly.

With consistently large linking numbers, it is also clear that the degeneracy of the Intrinsic Higgs sector grows very fast. For example, with linking numbers of mixed sizes, say $(a_1, a_2, a_3, a_4, a_5) = (2, 3, 5, 8, 13)$, we find a relatively small degeneracy,

$$\Omega_{\text{Higgs}}(y) \Big|_{\text{Intrinsic}}^{(2,3,5,8,13)} = 1261261/y + 1261261y . \tag{6.34}$$

With more consistently larger numbers, the growth of the Intrinsic Higgs sector is very rapid. For example, with the same notation as above, we have

$$\begin{aligned} \Omega_{\text{Higgs}}(y) \Big|_{\text{Intrinsic}}^{(3,5,7,9,11)} = & \\ & 54599524/y^9 + 3730179061/y^7 + 63638875882/y^5 + 379987985704/y^3 \\ & + 905199873928/y + 905199873928y + 379987985704y^3 \\ & + 63638875882y^5 + 373017906y^7 + 54599524y^9 , \end{aligned} \quad (6.35)$$

and

$$\begin{aligned} \Omega_{\text{Higgs}}(y) \Big|_{\text{Intrinsic}}^{(5,6,7,8,9)} = & \\ & 831775/y^{13} + 301581526/y^{11} + 22987872352/y^9 + 575641637000/y^7 \\ & + 5763858350669/y^5 + 25595480770735/y^3 + 53280763215115/y \\ & + 53280763215115y + 25595480770735y^3 + 5763858350669y^5 \\ & + 575641637000y^7 + 22987872352y^9 + 301581526y^{11} + 831775y^{13} , \end{aligned} \quad (6.36)$$

and as a final example,

$$\begin{aligned} \Omega_{\text{Higgs}}(y) \Big|_{\text{Intrinsic}}^{(8,9,10,11,12)} = & \\ & 32294250/y^{22} + 58872952926/y^{20} + 23086762587054/y^{18} \\ & + 3146301650299568/y^{16} + 186529800766285403/y^{14} \\ & + 5480846262397291070/y^{12} + 86780383421802203555/y^{10} \\ & + 783408269154731872224/y^8 + 4192271239441338802849/y^6 \\ & + 13657486692285216220742/y^4 + 27560691162972524163666/y^2 \\ & + 34791235315880411958041 + 27560691162972524163666y^2 \\ & + 13657486692285216220742y^4 + 4192271239441338802849y^6 \\ & + 783408269154731872224y^8 + 86780383421802203555y^{10} \\ & + 5480846262397291070y^{12} + 186529800766285403y^{14} \\ & + 3146301650299568y^{16} + 23086762587054y^{18} \\ & + 58872952926y^{20} + 32294250y^{22} . \end{aligned} \quad (6.37)$$

The growth is expected to be exponential, in general, as is appropriate for the interpretation of these states as black hole microstates.

6.3 Proof of the Second Conjecture

Now that we understand the general structure of $\Omega_{\text{Higgs}}(y)$ and its restriction to the Intrinsic Higgs sector, let us go ahead and prove the second conjecture in its full generality. Using the (already proven) first conjecture, invariance of the refined index of the Intrinsic Higgs sector amounts to k -independence of

$$\Omega_{\text{Higgs}}^{(k)}(y) - (-y)^{-d_k} D_k(-y) \quad \leftrightarrow \quad \Omega_{\text{Higgs}}^{(k)}(y) - \Omega_{\text{Coulomb}}^{(k)}(y) , \quad (6.38)$$

which can be also stated as the equality condition,

$$\Omega_{\text{Higgs}}^{(k)}(y) - \Omega_{\text{Higgs}}^{(k')}(y) = \Omega_{\text{Coulomb}}^{(k)}(y) - \Omega_{\text{Coulomb}}^{(k')}(y) \quad (6.39)$$

with arbitrary pairs of branches, k and k' , for any given quiver. In the following we will show that this latter statement holds for all cyclic Abelian quivers.

On the Higgs side, we start by rewriting (6.8) as contour integrals, so that $\Omega_{\text{Higgs}}^{(k)}(y) = (-y)^{-d_k} \chi_{\xi=-y^2}(M_k)$ is equal to

$$\frac{(-y)^{-d_k}}{(1-y^2)^n} \prod_{i \neq k} \oint_{J_i=0} \frac{dJ_i}{2\pi i} \left[\prod_{i \neq k} \left(\frac{1-y^2 e^{-J_i}}{1-e^{-J_i}} \right)^{a_i} \right] \cdot \left(\frac{1-e^{-\sum_{i \neq k} J_i}}{1-y^2 e^{-\sum_{i \neq k} J_i}} \right)^{a_k} , \quad (6.40)$$

which maps to, with $\omega_i \equiv e^{-J_i}$,

$$\frac{(-y)^{-d_k}}{(y^2-1)^n} \prod_{i \neq k} \oint_{\omega_i=1} \frac{d\omega_i}{2\pi i} \left[\prod_{i \neq k} \frac{1}{\omega_i} \left(\frac{1-y^2 \omega_i}{1-\omega_i} \right)^{a_i} \right] \cdot \left(\frac{1-\prod_{i \neq k} \omega_i}{1-y^2 \prod_{i \neq k} \omega_i} \right)^{a_k} . \quad (6.41)$$

A trick that simplifies the proof enormously is to rewrite the last factor of the integrand in terms of another contour integral with a dummy variable ω_k as

$$\begin{aligned} & \left(\frac{1-\prod_{i \neq k} \omega_i}{1-y^2 \prod_{i \neq k} \omega_i} \right)^{a_k} \\ &= \oint_{\omega_k=y^{-2} \prod_{i \neq k} \omega_i^{-1}} \frac{d\omega_k}{2\pi i} \left(\frac{1-y^2 \omega_k^{-1}}{1-\omega_k^{-1}} \right)^{a_k} \cdot \frac{1}{\omega_k - y^{-2} \prod_{i \neq k} \omega_i^{-1}} \\ &= -\frac{\prod_{i \neq k} \omega_i}{y^{2a_k-2}} \oint_{\omega_k=y^{-2} \prod_{i \neq k} \omega_i^{-1}} \frac{d\omega_k}{2\pi i} \left(\frac{1-y^2 \omega_k}{1-\omega_k} \right)^{a_k} \cdot \frac{1}{1-y^2 \prod_i \omega_i} \end{aligned} \quad (6.42)$$

The integrand has two additional poles at $\omega_k = 1$ and $\omega_k = \infty$, so we may trade this contour integral in favor of two others as,

$$= \frac{\prod_{i \neq k} \omega_i}{y^{2a_k-2}} \left(-\oint_{\omega_k=\infty} + \oint_{\omega_k=1} \right) \frac{d\omega_k}{2\pi i} \left(\frac{1-y^2 \omega_k}{1-\omega_k} \right)^{a_k} \cdot \frac{1}{1-y^2 \prod_i \omega_i}$$

$$= 1 + \frac{\prod_{i \neq k} \omega_i}{y^{2a_k-2}} \oint_{\omega_k=1} \frac{d\omega_k}{2\pi i} \left(\frac{1-y^2\omega_k}{1-\omega_k} \right)^{a_k} \cdot \frac{1}{1-y^2 \prod_i \omega_i} \quad (6.43)$$

where the first term “1” is from $\omega_k = \infty$ residue. Inserting this back into (6.41), we find $\Omega_{\text{Higgs}}^{(k)}(y)$ is composed of two additive pieces. The first piece, from “1”,

$$\frac{(-y)^{-d_k}}{(y^2-1)^n} \prod_{i \neq k} \oint_{\omega_i=1} \frac{d\omega_i}{2\pi i} \left[\prod_{i \neq k} \frac{1}{\omega_i} \left(\frac{1-y^2\omega_i}{1-\omega_i} \right)^{a_i} \right] \quad (6.44)$$

depends on the branch choice, k , while the second piece

$$\begin{aligned} & \frac{(-y)^{-d_k-2a_k+2}}{(y^2-1)^n} \prod_i \oint_{\omega_i=1} \frac{d\omega_i}{2\pi i} \left[\prod_i \left(\frac{1-y^2\omega_i}{1-\omega_i} \right)^{a_i} \right] \cdot \frac{1}{1-y^2 \prod_i \omega_i} \\ &= \frac{(-y)^{n+2-\sum_i a_i}}{(y^2-1)^n} \prod_i \oint_{\omega_i=1} \frac{d\omega_i}{2\pi i} \left[\prod_i \left(\frac{1-y^2\omega_i}{1-\omega_i} \right)^{a_i} \right] \cdot \frac{1}{1-y^2 \prod_i \omega_i} \end{aligned} \quad (6.45)$$

is manifestly independent of k .

The first, k -dependent, piece of $\Omega_{\text{Higgs}}^{(k)}(y)$ is explicitly integrated with

$$\oint_{\omega_i=1} \frac{d\omega_i}{2\pi i} \left(\frac{1-y^2\omega_i}{1-\omega_i} \right)^{a_k} \frac{1}{\omega_i} = y^{2a_i} - 1, \quad (6.46)$$

and becomes

$$\frac{(-y)^{-d_k}}{(y^2-1)^n} \prod_{i \neq k} (y^{2a_i} - 1) = (-1)^{d_k} y^{a_k} \prod_{i \neq k} \frac{y^{a_i} - y^{-a_i}}{y - y^{-1}}. \quad (6.47)$$

Thus, we have the general formula for the refined index in the Higgs phase for arbitrary cyclic Abelian quiver,

$$\begin{aligned} \Omega_{\text{Higgs}}^{(k)}(y) &= (-1)^{d_k} y^{a_k} \prod_{i \neq k} \frac{y^{a_i} - y^{-a_i}}{y - y^{-1}} \\ &+ \frac{(-y)^{n+2-\sum_i a_i}}{(y^2-1)^n} \prod_i \oint_{\omega_i=1} \frac{d\omega_i}{2\pi i} \left[\prod_i \left(\frac{1-y^2\omega_i}{1-\omega_i} \right)^{a_i} \right] \cdot \frac{1}{1-y^2 \prod_i \omega_i}. \end{aligned} \quad (6.48)$$

As a simple corollary, we have

$$\Omega_{\text{Higgs}}^{(k)}(y) - \Omega_{\text{Higgs}}^{(k')}(y) = (-1)^{d_k-1} \frac{y^{a_k-a_{k'}} - y^{-a_k+a_{k'}}}{y - y^{-1}} \prod_{i \neq k, k'} \frac{y^{a_i} - y^{-a_i}}{y - y^{-1}}, \quad (6.49)$$

where we remembered that $(-1)^{d_k} = (-1)^{-n+\sum_i a_i}$ is independent of k . We will presently compare this against the Coulomb phase counterpart.

For the Coulomb phase, we start with Eq. (4.17). Without loss of generality, we may suppose $a \equiv a_k - a_{k'} > 0$: If $a_k < a_{k'}$, we exchange the two labels, while, for $a_k = a_{k'}$, (4.17) shows $\Omega_{\text{Coulomb}}^{(k)}(y) - \Omega_{\text{Coulomb}}^{(k')}(y) = 0$ immediately, so no computation is needed. Taking the difference between branch k and branch k' , we find

$$\begin{aligned}
& G_k(y) - G_{k'}(y) \\
&= \sum_{\{t_{i \neq k, k'} = \pm 1\}} \left[\prod_{i \neq k, k'} t_i \right] \cdot \left[\Theta \left(\sum_{i \neq k, k'} a_i t_i - a \right) y^{\sum_{i \neq k, k'} a_i t_i - a} \right. \\
&\quad \left. - \Theta \left(\sum_{i \neq k, k'} a_i t_i + a \right) y^{\sum_{i \neq k, k'} a_i t_i + a} \right] \\
&= \sum_{\{t_{i \neq k, k'} = \pm 1\}} \left[\prod_{i \neq k, k'} t_i \right] \cdot \left[\Theta \left(\sum_{i \neq k, k'} a_i t_i - a \right) y^{\sum_{i \neq k, k'} a_i t_i} (y^{-a} - y^a) \right. \\
&\quad \left. - \left(1 - \Theta \left(\left| \sum_{i \neq k, k'} a_i t_i \right| - a \right) \right) y^{\sum_{i \neq k, k'} a_i t_i + a} - \delta_{\sum_{i \neq k, k'} a_i t_i + a, 0} \right]. \quad (6.50)
\end{aligned}$$

We perform a similar trick on $G(y^{-1})$'s and find

$$\begin{aligned}
& (-1)^n [G_k(y^{-1}) - G_{k'}(y^{-1})] \\
&= (-1)^n \sum_{\{t_{i \neq k, k'} = \pm 1\}} \left[\prod_{i \neq k, k'} t_i \right] \cdot \left[\Theta \left(\sum_{i \neq k, k'} a_i t_i - a \right) y^{-\sum_{i \neq k, k'} a_i t_i} (y^a - y^{-a}) \right. \\
&\quad \left. - \left(1 - \Theta \left(\left| \sum_{i \neq k, k'} a_i t_i \right| - a \right) \right) y^{-\sum_{i \neq k, k'} a_i t_i - a} - \delta_{-\sum_{i \neq k, k'} a_i t_i - a, 0} \right] \\
&= - \sum_{\{t_{i \neq k, k'} = \pm 1\}} \left[\prod_{i \neq k, k'} t_i \right] \cdot \left[\Theta \left(- \sum_{i \neq k, k'} a_i t_i - a \right) y^{\sum_{i \neq k, k'} a_i t_i} (y^a - y^{-a}) \right. \\
&\quad \left. - \left(1 - \Theta \left(\left| \sum_{i \neq k, k'} a_i t_i \right| - a \right) \right) y^{\sum_{i \neq k, k'} a_i t_i - a} - \delta_{\sum_{i \neq k, k'} a_i t_i - a, 0} \right], \quad (6.51)
\end{aligned}$$

where we have taken one more step of flipping the definition, $t_i \rightarrow -t_i$, at the last two lines.

Combining Eqs. (6.50) and (6.51) together, we find a vastly simplified expression,

$$\frac{G_k(y) + (-1)^n G_k(y^{-1}) - G_{k'}(y) - (-1)^n G_{k'}(y^{-1})}{(y - y^{-1})^n}$$

$$\begin{aligned}
&= (y - y^{-1})^{-n} (y^{-a} - y^a) \sum_{\{t_{i \neq k, k'} = \pm 1\}} \left[\prod_{i \neq k, k'} t_i \right] y^{\sum_{i \neq k, k'} a_i t_i} \\
&= -\frac{y^a - y^{-a}}{y - y^{-1}} \prod_{i \neq k, k'} \frac{y^{a_i} - y^{-a_i}}{y - y^{-1}}.
\end{aligned} \tag{6.52}$$

which is already finite at $y = 1$ so that the difference of the two counter polynomials vanishes on its own, $H_k(y) - H_{k'}(y) = 0$.

The difference of the equivariant Coulomb indices between branches k and k' is therefore

$$\Omega_{\text{Coulomb}}^{(k)}(y) - \Omega_{\text{Coulomb}}^{(k')}(y) = (-1)^{d_k - 1} \frac{y^{a_k - a_{k'}} - y^{-a_k + a_{k'}}}{y - y^{-1}} \prod_{i \neq k, k'} \frac{y^{a_i} - y^{-a_i}}{y - y^{-1}}. \tag{6.53}$$

Comparing this against (6.49), we see the two expressions are identical. This leaves behind only the case of $a_k = a_{k'}$, to which the above Coulomb phase procedure does not extend. However, as noted already, the difference vanishes in this case, which is mirrored by the Higgs phase result (6.49) as well. This establishes (6.39), which in turn guarantees that $\Omega_{\text{Higgs}}^{(k)}(y) - \Omega_{\text{Coulomb}}^{(k)}(y) = \Omega_{\text{Higgs}}^{(k)}(y) - (-y)^{-d_k} D_k(-y)$ is an invariant of the quiver. This generalizes the invariance proof in section 5 to the refined level.

7 Conclusion

In this note, we showed that the two conjectures proposed in Ref. [16] hold for all cyclic Abelian quivers, at a refined level with angular momentum and R -charge information: the Coulomb phase ground states are in one-to-one correspondence to elements of the pulled-back ambient cohomology, $i_{M_k}^*(H(X_k))$, while the remainder of $H(M_k)$, which we call the Intrinsic Higgs sector, is found to be insensitive to wall-crossing and defines an invariant of the quiver itself. Along the way, we constructed the refined index in the Higgs phase which computes the protected spin character of BPS states in four dimensions, and offered a routine for determining the entire Hodge diamond of the Higgs phase vacuum moduli space. Also found is a simple arithmetic formula for the Coulomb phase degeneracy.

As we already mentioned in section 2, all known BPS states, constructed to date from four-dimensional $N = 2$ field theories, are $SU(2)_R$ singlets. Partly based on such observations, it has been speculated that BPS states are neutral under R symmetry and thus are entirely classified by its angular momentum representations [18]. As far as we know, on the other hand, quivers that construct field theory BPS states admit no Intrinsic Higgs sector; it may as well be that $SU(2)_R$ singlet property is the

hallmark of the Coulomb phase states. The latter view is consistent with our finding $i_{M_k}^*(H(X_k)) = \bigoplus_p i_{M_k}^*(H^{p,p}(X_k))$ in all of our examples. For the Intrinsic Higgs states, however, vanishing R -charges would be strange, if not logically impossible; no classifying global charge would remain since they are inherently angular momentum singlets [16, 17].^{#11} Indeed, among several things we learned in this note is that, for the Intrinsic Higgs states, $U(1)'_R$ -charge are typically nontrivial, which ultimately means that these states are classified by $SU(2)_R$ of the underlying four-dimensional $N = 2$ theory.

This suggests a perhaps more physical, if less precise, criterion to separate the Coulomb states and the Intrinsic Higgs states (modulo those singlet states in $H^{d/2,d/2}$ for even d). The former states are all in the representation of type

$$[1/2 \text{ hyper}] \otimes (J, 0) , \quad (7.1)$$

while the latter states are in

$$[1/2 \text{ hyper}] \otimes (0, I) , \quad (7.2)$$

for some collection of J 's and I 's. We expect that $SU(2)_R$ representations are encoded entirely in \mathcal{I} eigenvalues and degeneracies thereof. Assuming single-center black hole interpretation of the latter, in particular, this implies that the microstates of BPS black holes of four-dimensional $N = 2$ theories are classified by $SU(2)_R$ multiplets. Finding explicit constructions of the corresponding black hole microstates from string theory models and comparing the resulting R -charge content against Eq. (6.7) should be most illuminating.

Although we have found an explicit and simple characterization that distinguish between Coulomb-like states and Intrinsic Higgs states, it remains a little mysterious from the Higgs phase viewpoint why the former states suffer wall-crossing while the latter do not. Physically, the single-center black hole interpretation for the Intrinsic Higgs states, as opposed to multi-center one, goes a long way explaining their invariance but then again, it is not exactly transparent why such a dichotomy of BPS states occurs and also why the field theory BPS states seemingly belong only to the former class of states.

At the level of quiver quantum mechanics, the disparity between the two phases can be understood from how ground state dynamics relate to the full quiver dynamics. For large absolute values of FI parameters, the Higgs phases are quite reliable as the vacuum manifold tends to be large and the truncated massive directions are very massive. For the Coulomb phase, things are a little more subtle, however. Small absolute

^{#11}The vanishing angular momentum as a criteria for single-center black hole states, as opposed to multi-center ones, was first proposed and tested extensively for 1/4 BPS black hole microstates in Refs. [25, 26].

values of FI constants favor this phase and wall-crossing physics become physically more transparent here, yet the naive truncation to the conventional Coulombic vacuum moduli space is dynamically unjustified due to small mass gaps along classically massive directions [13]. For index computation, this problem can be evaded via an index preserving deformation [13], but things become qualitatively more difficult for quivers that accept the so-called scaling solutions [12, 15]. The usual adoption of flat kinetic term is no longer justified, even for the purpose of index computations, near the origin where two or more charge centers approach each other arbitrarily close. There, the topology can be quite different than naively assumed [14], casting some doubt on the usual prescription. Such subtleties may explain why the Coulomb phase fails to capture the entire low energy aspects of quiver dynamics.

On a more mathematical side, we can also ask how this relates to the proposal of Kontsevich and Soibelman, who offered a simple algebraic structure that is supposed to capture the wall-crossing behavior in a universal manner [27, 28, 29]. In this approach, indices on two sides of a given marginal stability wall enters, respectively, a string of operator product as exponents. Wall-crossing data is recovered, then, by demanding that two such operator products equal each other, which constrains exponents of one side given those on other side. The equivalence of this proposal with physically derived wall-crossing formulae has been tested extensively, but only for examples where the Intrinsic Higgs sectors are absent.^{#12} This leads to the question of whether and how the information of the Intrinsic Higgs sector enters this algebraic formulation. While the underlying multi-center physics of wall-crossing is very clearly encoded in the Coulomb phase, the universal nature of the algebra suggests that the total index rather than just the Coulomb phase index would enter the algebra. Whether this is true or not needs to be confirmed, to begin with. Assuming an affirmative answer, it would also raise a question of whether and how Kontsevich-Soibelman algebra might know about the quiver invariants in some natural manner.

Apart from the next obvious task of verifying these conjectures for general quivers, also of some interest is generalization of this story to 1/4 BPS states of four-dimensional $N = 4$ theories [30]. In fact, the multi-centered nature and the intuitive understanding of wall-crossing of what we now call Coulomb phase states was first discovered in the context of such states [5, 6, 31], and quiver representations are also available for them, albeit with complications from having adjoint Higgs fields. Analogs of the refined index for these 1/4 BPS states have been identified recently [32], which may be explored along the same line as here.

Acknowledgments

^{#12}See Ref. [11] for the most complete comparison, to date.

We are grateful to Ashoke Sen for useful comments and indebted to HwanChul Yoo for helpful advice on combinatorics issues. This work is supported by the National Research foundation of Korea (NRF) funded by the Ministry of Education, Science and Technology with the grant number 2010-0013526 and also in part (PY) via the Center for Quantum Spacetime with grant number 2005-0049409.

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